Algorithmic Randomness of Continuous Functions

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Abstract

We investigate notions of randomness in the space $\mathcal{C}(2^{\mathbb{N}})$ of continuous functions on $2^{\mathbb{N}}$. A probability measure is given and a version of the Martin-Löf Test for randomness is defined. Random Δ^0_2 continuous functions exist, but no computable function can be random and no random function can map a computable real to a computable real. The image of a random continuous function is always a perfect set and hence uncountable. For any $y \in 2^{\mathbb{N}}$, there exists a random continuous function F with y in the image of F. Thus the image of a random continuous function need not be a random closed set. The set of zeroes of a random continuous function is always a random closed set.

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1 Introduction

The study of algorithmic randomness has been of great interest in recent years. The basic problem is to quantify the randomness of a single real number. Early in the last century, von Mises [26] suggested that a random real should obey reasonable statistical tests, such as having a roughly equal number of zeroes and ones of the first n bits, in the limit. Thus a random real would be *stochastic* in modern parlance. If one considers only *computable* tests, then there are countably many such tests and one can construct a real satisfying all tests.

Martin-Löf [20] observed that stochastic properties could be viewed as special kinds of measure zero sets and defined a random real as one which avoids certain effectively presented measure 0 sets. That is, a real $x \in 2^{\mathbb{N}}$ is Martin-Löf random if for any effective sequence $S_1, S_2, \ldots$ of c.e. open sets with $\mu(S_n) \leq 2^{-n}$, $x \notin \cap_n S_n$.

At the same time Kolmogorov [15] defined a notion of randomness for finite strings based on the concept of *incompressibility.* For infinite words, the stronger notion of prefix-free complexity developed by Levin [19], Gács [13] and Chaitin [8] is needed. Schnorr later proved that the notions of Martin-Löf randomness and Chaitin randomness are equivalent.

In a recent paper [2], the notion of randomness was extended to finite-branching trees and effectively closed sets. It was shown that a random closed set is perfect and contains no computable elements (in fact, it contains no n-c.e. elements). Every random closed set has measure 0 and has Hausdorff dimension $log_2 \frac{4}{3}$.

In this paper we want to consider algorithmic randomness on the space $\mathcal{C}(2^{\mathbb{N}})$ of continuous functions $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$.

Some definitions are needed. For a finite string $\sigma \in \{0,1\}^n$, let $|\sigma| = n$. For two strings σ, τ, say that τ extends σ and write $\sigma \prec \tau$ if $|\sigma| \leq |\tau|$ and $\sigma(i) = \tau(i)$ for $i < |\sigma|$. Similarly $\sigma \prec x$ for $x \in 2^{\mathbb{N}}$ means that $\sigma(i) = x(i)$ for $i < |\sigma|$. Let $\sigma^\frown\tau$ denote the concatenation of σ and τ and let $\sigma^\frown i$ denote $\sigma^\frown(i)$ for $i = 0, 1$. Let $x\lceil n = (x(0), \ldots, x(n-1))$. Two reals x and y may be coded together into $z = x \oplus y$, where $z(2n) = x(n)$ and $z(2n+1) = y(n)$ for all n.

For a finite string σ, let $I(\sigma)$ denote $\{x \in 2^{\mathbb{N}} : \sigma \prec x\}$. We shall call $I(\sigma)$, the *interval* determined by σ. Each such interval is a clopen set and the clopen sets are just finite unions of intervals. We let $\mathcal{B}$ denote the Boolean algebra of clopen sets.

Now a nonempty closed set P may be identified with a tree $T_P \subseteq \{0,1\}^*$ where $T_P = \{\sigma : P \cap I(\sigma) \neq \emptyset\}$. Note that T_P has no dead ends. That is, if $\sigma \in T_P$, then either $\sigma^\frown 0 \in T_P$ or $\sigma^\frown 1 \in T_P$.

For an arbitrary tree $T \subseteq \{0,1\}^*$, let $[T]$ denote the set of infinite paths through T, that is,

$$x \in [T] \iff (\forall n) x\lceil n \in T.$$

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It is well-known that $P \subseteq 2^{\mathbb{N}}$ is a closed set if and only if $P = [T]$ for some tree T. P is a Π^0_1 class, or an effectively closed set, if $P = [T]$ for some computable tree T. P is a strong Π^0_2 class, or a Π^0_2 closed set, if $P = [T]$ for some Δ^0_2 tree. The complement of a Π^0_1 class is sometimes called a c.e. open set. We remark that if P is a Π^0_1 class, then T_P is a Π^0_1 set, but it is not, in general, computable. There is a natural effective enumeration $P_0, P_1, \ldots$ of the Π^0_1 classes and thus an enumeration of the c.e. open sets. Thus we can say that a sequence $S_0, S_1, \ldots$ of c.e. open sets is *effective* if there is a computable function, f, such that $S_n = 2^{\mathbb{N}} - P_{f(n)}$ for all n. For a detailed development of Π^0_1 classes, see [5, 6].

The betting approach to randomness is formalized as follows:

Definition 1.1 (Ville [25]). (i) *A* martingale *is a function* $d : n^{<\mathbb{N}} \to [0, \infty)$ *such that for all* $\sigma \in n^{<\mathbb{N}}$*,*

$$d(\sigma) = \frac{1}{n} \sum_{i=0}^{n-1} d(\sigma^\frown i).$$

(ii) *A martingale* d succeeds *on* $X \in n^{\mathbb{N}}$ *if*

$$\limsup_{m \to \infty} d(X \lceil m) = \infty.$$

That is, the betting strategy results in an unbounded amount of money made on the binary string X*.*

(iii) *The* success set *of* d *is the set* $S^\infty[d]$ *of all sequences on which* d *succeeds.*

That is, a martingale on $2^{<\mathbb{N}}$ is the representation of a fair double-or-nothing betting strategy. When working on $3^{<\mathbb{N}}$ the strategy is triple-or-nothing.

Definition 1.2. *A martingale* d *is* constructive (effective, c.e.) *if it is* lower semi-computable*; that is, if there is a computable function* $\hat{d} : n^{<\mathbb{N}} \times \mathbb{N} \to \mathbb{Q}$ *such that*

(i) *for all* σ *and* t, $\hat{d}(\sigma, t) \leq \hat{d}(\sigma, t + 1) < d(\sigma)$*, and*

(ii) *for all* σ, $\lim_{t \to \infty} \hat{d}(\sigma, t) = d(\sigma)$*.*

In other words, $d(w)$ is approximated from below by rationals uniformly in w. A sequence in $2^{\mathbb{N}}$ is considered random in this setting if no constructive martingale succeeds on it.

Martin-Löf randomness for reals, as defined above, is extended to closed sets by giving an effective homeomorphism with the space $\{0, 1, 2\}^{\mathbb{N}}$ and simply carrying over the notion of randomness from that space. A continuous function F may be represented by an element of $\{0, 1, 2\}^{\mathbb{N}}$ and is said to be Martin-Löf random if it has a random representation.

2 Random closed sets and continuous functions

We will define the notion of a random continuous function along similar lines to the definition of a random closed set in [2]. The definition of a random (nonempty) closed set $P = [T]$ (where $T = T_P$) comes from a probability measure μ^* where, given a node $\sigma \in T$, each of the following scenarios has equal probability $\frac{1}{3}$:

$\sigma^\frown 0 \in T$ and $\sigma^\frown 1 \in T$,

$\sigma^\frown 0 \in T$ and $\sigma^\frown 1 \notin T$, and

$\sigma^\frown 0 \notin T$ and $\sigma^\frown 1 \in T$.

More formally, we define a measure μ^* on the space $\mathcal{C}$ of closed subsets of $2^{\mathbb{N}}$ as follows. Given a closed set $Q \subseteq 2^{\mathbb{N}}$, let $T = T_Q$ be the tree without dead ends such that $Q = [T]$. Let $\sigma_0, \sigma_1, \ldots$ enumerate the elements of T in order, first by length and then lexicographically. We then define the code $x = x_Q = x_T$ by recursion such that for each n, $x(n) = 2$ if both $\sigma_n{}^\frown 0$ and $\sigma_n{}^\frown 1$ are in T, $x(n) = 1$ if $\sigma_n{}^\frown 0 \notin T$ and $\sigma_n{}^\frown 1 \in T$, and $x(n) = 0$ if $\sigma_n{}^\frown 0 \in T$ and $\sigma_n{}^\frown 1 \notin T$. We then define a measure μ^* on $\mathcal{C}$ by setting

$$\mu^*(\mathcal{X}) = \mu(\{x_Q : Q \in \mathcal{X}\}) \tag{1}$$

for $\mathcal{X} \subseteq \mathcal{C}$, where μ is the standard measure on $\{0,1,2\}^{\mathbb{N}}$. Then Brodhead, Cenzer, and Dashti [2] defined a a closed set $Q \subseteq 2^{\mathbb{N}}$ to be (Martin-Löf) random if x_Q is (Martin-Löf) random.

A continuous function on $2^{\mathbb{N}}$ is a function with a closed graph. Thus we might simply say that a function F is random if the graph $Gr(F)$ is a random closed set. Now $Gr(F) = \{x \oplus y : y = F(x)\}$. Thus if $[T]$ is the graph of a function and $\sigma \in T$ has even length, then we must have $\sigma^\frown 0 \in T$ and $\sigma^\frown 1 \in T$. This means that the family of closed sets which are the graphs of functions has measure 0 in the space of closed sets and hence a random closed set will not be the graph of a function. So we need a different measure to define randomness for continuous functions.

For any continuous function F on $2^{\mathbb{N}}$ and any $\sigma \in \{0,1\}^*$, there is a natural number n and binary string τ of length n such that for all $u \in I(\sigma)$, $F(u)\lceil n = \tau$. In particular, $F(u)(n) = \tau(n)$ for every such u. In general, the length of σ may be much larger than n, so we may have to extend σ by several bits to get uniformity of $F(u)\lceil(n+1)$ within the interval around σ's extension. Thus we recursively define a computation tree on $\{0,1\}^*$ for F by attaching a *label* $f(\sigma) \in \{0,1,2\}$ to each node, as follows. The root node $\emptyset$ is left unlabeled. For $|\sigma| = m+1$, having defined $f(\sigma\lceil i) = e_i$ for all $i \leq m$, let $\rho = (n_1, \ldots, n_k)$ be the result of deleting all 2s from $(e_1, \ldots, e_m)$. If for all $u \in I(\sigma)$, $F(u)\lceil k = \rho^\frown j$, $j \in \{0,1\}$, we may let $e_{m+1} = j$. If not we must have $e_{m+1} = 2$; even if so we allow $e_{m+1} = 2$. Thus for any continuous F there exist infinitely many representing functions $f : \{0,1\}^* \to \{0,1,2\}$. The representation which uses as few 2s as possible we shall call the *canonical representation*. Finally, we want to code the representing function as an element of $3^{\mathbb{N}}$ to discuss its algorithmic randomness. Enumerate $\{0,1\}^* = \{\emptyset\}$ as $\sigma_0, \sigma_1, \ldots$, ordered first by length and then lexicographically. Thus $\sigma_0 = (0)$, $\sigma_1 = (1)$, $\sigma_2 = (00)$, etc.

Definition 2.1. (i) *Let INF equal the set of $y \in \{0,1,2\}^n$ such that $\{n : y(n) \neq 2\}$ is infinite and, for $y \in INF$, let $G(y)$ be the result of removing from x all occurrences of 2.*

(ii) *A function $f : \{0,1\}^* \to \{0,1,2\}$* represents *a function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ if for all $x \in 2^{\mathbb{N}}$, the sequence y, defined by $y(n) = f(x\lceil n)$ belongs to INF and $G(y) = F(x)$.*

(iii) *A sequence $r \in \{0,1,2\}^{\mathbb{N}}$* represents *the continuous function F (written $F = F_r$) if the function $f_r : \{0,1\}^* \to \{0,1,2\}$, defined by $f_r(\sigma_n) = r(n)$, represents F.*

This representation may be given by a *labelled* tree, where the value $f(\sigma)$ is attached to the each node $\sigma \in \{0,1\}^*$. For example, the identity function can be represented by placing an e on any node σ which ends in e. This can also be pictured geometrically as representing the graph of F as the intersection of a decreasing sequence of clopen subsets of the unit square. Initially the choice of $f((0))$ and $f((1))$ selects from the 4 quadrants. That is, for example, $f((0)) = (0) = f((1))$ implies that the graph of F is included in the bottom half of the square and $f((0)) = \emptyset$ and $f((1)) = (1)$ implies that the graph excludes the lower right hand quadrant. Successive values of f continue to restrict the graph of F in a similar fashion.

Randomness for continuous functions is defined by using the Lebesgue measure on the space $3^{\mathbb{N}}$ of representations. Thus for each new bit of input, there is equal probability $\frac{1}{3}$ that f_r gives a new output of 0 for F_r, gives a new output of 1 for F_r, or gives no new output for F_r. This will induce a measure μ^{**} on the space $\mathcal{F}$ of continuous functions.

Definition 2.2. *A function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is* random *if there is a sequence $r \in 3^{\mathbb{N}}$ such that r is random with respect to the measure μ^{**}.*

Our first result will take care of the functions f which do not represent a total function. The following lemma is needed.

Lemma 2.3. *Let Σ be a finite set and let $Q \subseteq \Sigma^{\mathbb{N}}$ be a Π^0_1 class of measure 0. Then no element of Q is Martin-Löf random.*

Proof. Let $\Sigma = \{0,1,2\}$ without loss of generality. Let $Q = [T]$ where $T \subseteq \{0,1,2\}^*$ is a computable tree (possibly with dead ends). For each n, let $T_n = T \cap \{0,1,2\}^n$ and let

$$Q_n = \bigcup \{I(\sigma) : \sigma \in T_n\}.$$

Let $g(n) = \mu(Q_n) = \frac{|T_n|}{3^n}$. Then $g(n)$ is a computable sequence and

$$lim_{n \to \infty} g(n) = \mu(Q) = 0.$$

This Martin-Löf test shows that Q has no random elements. (As observed by Solovay, it is sufficient to have a computable sequence approaching zero rather than the stricter test with a sequence of measures $g(n) \leq 2^{-n}$.) $\square$

Theorem 2.4. *The set of functions in $3^{\mathbb{N}}$ which represent a total continuous function has measure one, and every random function represents a continuous function.*

Proof. Let $f \in 3^{\mathbb{N}}$ and suppose that f does not represent a total function. Then there is some $x \in 2^{\mathbb{N}}$ and some $\tau \in \{0,1\}^*$ such that $f(x\lceil n) = \tau$ for almost all n. Without loss of generality we may assume that $\tau = \emptyset$. Let A be the set of functions $f : \{0,1\}^* \to \{0,1\}^*$ such that $f(\sigma) = \emptyset$ for arbitrarily long strings σ and let $p = \mu^{**}(A)$. Then certainly $p \leq \frac{5}{9}$, since if $r(0)$ and $r(1)$ are both in $\{0,1\}$, then $f_r \notin A$. Considering the 9 cases for the initial choices of $f((0))$ and $f((1))$, we see that

$$p = \frac{4}{9}p + \frac{1}{9}[1 - (1-p)^2],$$

so that $\frac{1}{9}p^2 + \frac{1}{3}p = 0$, which implies that $p = 0$. (That is, there are 4 cases in which $|f((i))| = 1$ for $i = 0, 1$ so that immediately $f \notin A$, there are 4 cases in which only one of $f((i)) = \emptyset$, in which case the remaining function g, defined by $g(\sigma) = f(i^\frown\sigma)$ must be in A, and there is one case in which $f((i)) = \emptyset$ for $i = 0, 1$, in which case at least one of the remaining functions must be in A.)

Observe that A is a Π^0_1 class, since $f_r \in A$ if and only if $(\forall n)(\exists\sigma \in \{0,1\}^n) f_r(\sigma) = \emptyset$. It follows from Lemma 2.3 that no random function can be in A and therefore every random function $f : \{0,1\}^* \to \{0,1\}^*$ indeed represents a continuous function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$. $\square$

Now the set of Martin-Löf random elements of $\{0,1,2\}^{\mathbb{N}}$ has measure one and there exists a Δ^0_2 Martin-Löf real. Hence we have the following.

Theorem 2.5. *There exists a random continuous function which is Δ^0_2 computable.*

Next we obtain some properties of random continuous functions.

We first observe that any continuous function will have a representation which is not random. In fact, the canonical representation itself can never be random.

Proposition 2.6. *For any continuous function F, the canonical representation is not random.*

Proof. The idea is that whenever the canonical representation labels a node σ with 2, then the two labels on the successor nodes $\sigma^\frown 0$ and $\sigma^\frown 1$ cannot be both 0, or both 1. Thus we have the following Martin-Löf test. Assume by way of contradiction that r is random and canonical. Let S_e be the set of $r \in 3^{\mathbb{N}}$ such that r has at least e occurrences of 2 and such that, for the first e occurrences of 2 in r, the corresponding successor values are not both 0 or both 1. Since r is random, it must have infinitely many occurrences of 2 and since r is canonical, it must belong to every S_e. But each S_e is a c.e. open set and has measure $\leq (\frac{7}{9})^e$, so that no random sequence can belong to every S_e. □

For any function F on $2^{\mathbb{N}}$ and any $\sigma \in \{0,1\}^*$, define the restriction F_σ of F to $I(\sigma)$ by

$$F_\sigma(x) = F(\sigma^\frown x).$$

Clearly any such restriction of a random continuous function will be random, but more can be said.

First recall van Lambalgen's theorem.

Theorem 2.7 (van Lambalgen [24]). *The following are equivalent.*

1. *$A \oplus B$ is n-random.*
2. *A is n-random and B is n-A-random (or vice-versa).*
3. *A is n-B-random and B is n-A-random.*

Proposition 2.8. *F is a random continuous function if and only if the functions $F_{(0)}$ and $F_{(1)}$ are relatively random.*

Proof. Let r represent F. Suppose first that F is random. It follows as in the proof of Lemma 2.6 of [3] that $F_{(0)} \oplus F_{(1)}$ is random and hence $F_{(0)}$ and $F_{(1)}$ are relatively random by van Lambalgen's theorem.

Next suppose that $F_{(0)}$ and $F_{(1)}$ are relatively random and let r_i represent $F_{(i)}$ for $i = 0,1$. Let d be any martingale, which we think of as betting on r. Then for $i = 0,1$, we can define a martingale d_i with oracle r_{1-i} as follows. We will give the definition for d_0 and leave d_1 for the reader. Given $\sigma = r_0(0),\dots,r_0(2^p+q-2)$ where $0 \leq q < 2^p$, use r_1 to compute $\tau = r(0),\dots,r(2^{p+1}+q-2)$ and then define d_i to bet in the same proportion as d. That is, $d_i(\sigma^\frown j)/d_i(\sigma) = d(\tau^\frown j)/d(\tau)$ for $j < 3$. Thus for any node on the left side of the labelled tree for F, d_0 is making the same bet on the next label that d would have made, and similarly for d_1 and the right side.

Since the $F_{(i)}$ are relatively random for $i = 0,1$, it follows that d_i does not succeed and hence there exist upper bounds B_i for $\{d_i(r_i\lceil n)\}_{n\in\mathbb{N}}$. But it follows from the above definitions of d_i that for any p,

$$d(r\lceil 2^{p+1}-2) = d_0(r_0\lceil 2^p - 1)\cdot d_1(r_1\lceil 2^p - 1).$$

This is because the martingale d alternates using d_0 and d_1 and the result can be viewed in each alternation as multiplying the capital by some factor. Then in general, for $0 < q \leq 2^p$,

$$d(r\lceil 2^{p+1}+q-2) = d_0(r_0\lceil 2^p+q-1)\cdot d_1(r_1\lceil 2^p - 1)$$

and

$$d(r\lceil 2^{p+1}+2^p+q-2) = d_0(r_0\lceil 2^{p+1}-1)\cdot d_1(r_1\lceil 2^p+q-1).$$

It follows that $B_0 \cdot B_1$ is an upper bound for $\{d(r\lceil k) : k \in \mathbb{N}\}$, so that d does not succeed on r. □

Proposition 2.9. *Suppose $A \subset B$ are two finite sets of symbols. Given $X \in B^{\mathbb{N}}$, let $\widetilde{X} \in A^{\mathbb{N}}$ be the sequence obtained by deleting all symbols in $B - A$ from X. If X is 1-random, then $\widetilde{X}$ is 1-random.*

Proof. Given $X, \widetilde{X}$ as in the proposition, suppose $\widetilde{X}$ is not random and let d be a constructive martingale on $A^{\mathbb{N}}$ that succeeds on $\widetilde{X}$. We will construct a martingale $\hat{d}$ on $B^{\mathbb{N}}$ that succeeds on $\widetilde{X}$. Essentially, $\hat{d}$ will keep its capital constant on symbols in $B - A$; it will bet according to d, repeating its bets after bits which hold symbols from $B - A$.

Define $\hat{d}(\lambda) = d(\lambda)$, and for $\sigma \in B^*$ and $\tilde{\sigma}$ the corresponding string of A^*,

$$\hat{d}(\sigma^\frown x) = \begin{cases} \frac{d(\tilde{\sigma}^\frown x)}{d(\tilde{\sigma})}\hat{d}(\sigma) & x \in A \\ \hat{d}(\sigma) & x \in B - A \end{cases}$$

The function $\hat{d}$ is clearly constructive, since d is. To show $\hat{d}$ is a martingale, consider the sum

$$\sum_{x \in B} d(\sigma^\frown x) = \sum_{x \in A} \frac{d(\tilde{\sigma}^\frown x)}{d(\tilde{\sigma})}\hat{d}(\sigma) + \sum_{x \in B - A} \hat{d}(\sigma)$$

$$= \hat{d}(\sigma) \sum_{x \in A} \frac{d(\tilde{\sigma}^\frown x)}{d(\tilde{\sigma})} + \hat{d}(\sigma)|B - A| = \hat{d}(\sigma)[|A| + |B - A|].$$

It remains to show that $\hat{d}$ succeeds on X. However, that is clear, as on bits which are in X but not $\widetilde{X}$, $\hat{d}$ keeps its capital constant, and on bits from $\widetilde{X}$, it acts exactly as d would. Therefore since d succeeds on $\widetilde{X}$, $\hat{d}$ succeeds on X and X is nonrandom. □

It is easy to see that, for any random continuous function F and any computable real x, $F(x)$ is not computable. This also follows from our next result.

Theorem 2.10. *If F is a random continuous function, then, for any computable real x, $F(x)$ is a random real.*

Proof. Suppose that F is random with representing function f_r, let x be a computable real and let $y = F(x)$. Define the computable function g so that, for each n,

$$\sigma_{g(n)} = x \lceil n.$$

By the Von-Mises–Church–Wald Computable Selection Theorem, the subsequence $z(n) = r(g(n))$ is random in $\{0, 1, 2\}^{\mathbb{N}}$. Now $y = F(x)$ may be computed from z by removing the 2's. Thus $F(x)$ is random by Proposition 2.9. □

We note that Fouche [12] has used a different approach to randomness for continuous functions connected with Brownian motion, first presented by Asarin and Prokovskiy [1], and has shown that, under this approach, it is also true that for any random continuous function F, $F(x)$ is not computable for any computable input x.

It follows that a random function F can never be computably continuous and hence the graph of F is not a Π^0_1 class.

Theorem 2.11. *If F is a random continuous function, then the image $F[2^{\mathbb{N}}]$ has no isolated elements.*

Proof. Let f be the random representing function for F and let $Q = F[2^{\mathbb{N}}]$. Suppose by way of contradiction that Q contains an isolated path y. Then there is some finite $\tau \prec y$ such that y is the unique element of $I(\tau) \cap Q$. Fix σ such that $f(\sigma) = \tau$.

For each n, let S_n be the set of all $g \in \mathcal{F}$ such that for all $\rho_1, \rho_2 \in \{0,1\}^n$,

1. $g(\sigma^\frown \rho_1)$ is compatible with $g(\sigma^\frown \rho_2)$,
2. $\tau \prec g(\sigma^\frown \rho_1)$, and
3. $\tau \prec g(\sigma^\frown \rho_2)$

Then for any each $m < n$ and each $\rho \in \{0,1\}^m$, we are restricted to at most 7 of the 9 possible choices for $f(\rho^\frown 0)$ and $f(\rho^\frown 1)$. This same scenario applies for all $\rho \in \{0,1\}^{n-1}$, so that in general, $\mu(S_n) \leq (\frac{7}{9})^{2^{n-1}}$.

Now for each n, S_n is a clopen set in $\mathcal{F}$ and thus the sequence $S_0, S_1, \ldots$ is a Martin-Löf test. It follows that for some n, $F \notin S_n$. Thus there are two extensions of σ of length n which have incompatible images, contradicting the assumption that y was the unique element of $Q \cap I(\tau)$. □

It follows that the image of a random continuous function is perfect and has continuum many elements. There are several natural questions about the image $F[2^{\mathbb{N}}]$ of a random continuous function F. Is the image of F a random closed set? What is the measure of the image? Can the function be onto? We will give some partial answers.

It follows from Proposition 2.8 that, for any $\tau \in \{0,1\}^*$, there is a random continuous function with image $\subseteq I(\tau)$. Thus a random continuous function is not necessarily onto.

Theorem 2.12. *For any $\sigma \in \{0,1\}^*$, the probability that the image of a continuous function F meets $I(\sigma)$ is always $> \frac{3}{4}$.*

Proof. The proof is by induction on $|\sigma|$. Without loss of generality, we assume that $\sigma = 0^n$. For each $n > 0$, let q_n be the probability that $F[2^{\mathbb{N}}]$ meets $I((0^n))$. Let f be the representing function for F. For $n = 1$, there are 9 equally probable choices for the pair $f((0))$ and $f((1))$, breaking down into 4 distinct cases.

Case 1. If $f((0)) = (1) = f((1))$, then $F[2^{\mathbb{N}}]$ does not meet $I((0))$. This occurs just once.

Case 2. If $f((0)) = (0)$ or $f((1)) = (0)$, then $F[2^{\mathbb{N}}]$ meets $I((0))$. This occurs in 5 of the 9 choices.

Case 3. If $f((i)) = \emptyset$ and $f((1-i)) = (1)$, then $F[2^{\mathbb{N}}]$ meets $I((0))$ if and only if $F_{(i)}[2^{\mathbb{N}}]$ meets $I((0))$. This occurs in 2 of the 9 choices, with probability q_1.

Case 4. If $f((0)) = \emptyset = f((1))$, then $F[2^{\mathbb{N}}]$ meets $I((0))$ if at least one of $F_{(i)}[2^{\mathbb{N}}]$ meets $I((0))$. This occurs in 1 of the choices, with probability $1-(1-q_1)^2$. That is, $F[2^{\mathbb{N}}]$ *fails to meet* $I((0))$ if *both* $F_{(0)}[2^{\mathbb{N}}]$ *and* $F_{(0)}[2^{\mathbb{N}}]$ fail to meet $I((0))$.

Putting these cases together, we see that

$$q_1 = \frac{5}{9} + \frac{2}{9}q_1 + \frac{1}{9}(2q_1 - q_1^2),$$

so that q_1 satisfies the quadratic equation

$$x^2 + 5x - 5 = 0.$$

Thus q_1 is the unique solution in [0,1] of this equation, that is,

$$q_1 = \frac{\sqrt{45}-5}{2},$$

which is indeed $> .75$.

Now let $q_n = q$ and let $q_{n+1} = p$. Once again we consider the 9 initial choices, now breaking down into 6 distinct cases.

Case 1. If $f((0)) = (1) = f((1))$, then $F[2^{\mathbb{N}}]$ does not meet $I((0^{n+1}))$. This occurs just once.

Case 2. If $f((0)) = (0) = f((1))$, then $F[2^{\mathbb{N}}]$ meets $I((0^{n+1}))$ if and only if at least one of $F_{(0)}$ and $F_{(1)}$ meets $I((0^n))$. This occurs just once, and with probability $1-(1-q)^2 = 2q - q^2$.

Case 3. If $f((i)) = (0)$ and $f((1-i)) = (1)$, then $F[2^{\mathbb{N}}]$ meets $I((0^{n+1}))$ if and only if $F_{(i)}[2^{\mathbb{N}}]$ meets $I((0^n))$. This occurs in 2 of the 9 choices, with probability q.

Case 4. If $f((i)) = \emptyset$ and $f((1-i)) = (1)$, then $F[2^{\mathbb{N}}]$ meets $I((0^{n+1}))$ if and only if $F_{(i)}[2^{\mathbb{N}}]$ meets $I((0^{n+1}))$. This occurs in 2 of the 9 choices, with probability p.

Case 5. If $f((0)) = \emptyset = f((1))$, then $F[2^{\mathbb{N}}]$ meets $I((0^{n+1}))$ if at least one of $F_{(i)}[2^{\mathbb{N}}]$ meets $I((0^{n+1}))$. This occurs just once, with probability $1-(1-p)^2$.

Case 6. If $f((i)) = \emptyset$ and $f((1-i)) = (0)$, then $F[2^{\mathbb{N}}]$ meets $I((0^{n+1}))$ if at least one of the following two things happens. Either $F_{(i)}[2^{\mathbb{N}}]$ meets $I((0^{n+1}))$, or $F_{(1-i)}[2^{\mathbb{N}}]$ meets $I((0^n))$. This occurs in 2 of the 9 choices, with probability $1-(1-p)(1-q)$.

Putting these cases together, we see that

$$p = \frac{2}{3}p - \frac{1}{9}p^2 - \frac{2}{9}pq + \frac{2}{3}q - \frac{1}{9}q^2,$$

so that $p = q_{n+1}$ satisfies the equation

$$p^2 + 3p + 2pq - 6q + q^2 = 0.$$

We note that for $p = q$, the solutions are $p = q = 0$ and $p = q = \frac{3}{4}$. This explains the value $\frac{3}{4}$ in the statement of theorem.

Now assume by induction that $q > \frac{3}{4}$. Suppose by way of contradiction that $p \leq \frac{3}{4}$. It follows that

$$\frac{9}{16} + \frac{9}{4} + \frac{3}{2}q - 6q + q^2 \geq 0.$$

Simplifying, this implies that $16q^2 - 72q + 45 \geq 0$. But this factors into $(4q - 3)(4q - 15)$ and is only ≥ 0 when either $q \leq \frac{3}{4}$ or $q \geq \frac{15}{4}$. Since the latter is impossible, we obtain the desired contradiction that $q \leq \frac{3}{4}$. ☐

Corollary 2.13. *For any $y \in 2^{\mathbb{N}}$,*

(a) $\mu^{**}(\{F : y \in F[2^{\mathbb{N}}]\}) = \frac{3}{4}$;

(b) there exists a random continuous function F with $y \in F[2^{\mathbb{N}}]$.

Proof. (a) Let p be the probability that $y \in F[2^{\mathbb{N}}]$. It follows that for each $\sigma \in \{0,1\}^n$, the probability that $y \in F[I(\sigma)]$, given that $f(\sigma)$ is consistent with y, also equals p. It follows from the proof of Theorem 2.12 that $p = \frac{3}{4}$.

(b) Since the random continuous functions have measure 1 in $\mathcal{C}(2^{\mathbb{N}})$, it follows that some random continuous function has y in the image. ☐

Corollary 2.14. *The image of a random continuous function need not be a random closed set.*

Proof. It was shown in [2] that a random closed set has no computable members. Let F be a random continuous function with 0^ω in the image, as given by Corollary 2.13. Then $F[2^{\mathbb{N}}]$ is not a random closed set. ☐

3 Zeroes of Random Continuous Functions

In this section we prove that for any random continuous function F, the set $\mathcal{Z}(F) = \{x : F(x) = 0\}$ is a random closed set. For any subset S of $\mathcal{C}$, let $Z_S = \{F \in \mathcal{F} : Z(F) \in S\}$.

Lemma 3.1. *For any open set S, $\mu^{**}(Z_S) \leq \mu^*(S)$.*

Proof. It suffices to prove the result for intervals $S = I(\sigma)$. We will show by induction on $|\sigma|$ that $\mu^{**}(Z_{I(\sigma)}) = (\frac{1}{4})^{|\sigma|}$, whereas of course $\mu^*(I(\sigma)) = (\frac{1}{3})^{|\sigma|}$. Recall from Corollary 2.13 that $0 \in F[2^{\mathbb{N}}]$ with probability exactly $\frac{3}{4}$. For $|\sigma| = 1$, there are two distinct cases.

Case I Suppose first that $\sigma = (i)$, where $i \in \{0,1\}$. Then $F \in Z_S$ if and only if F has a zero in $I((i))$ and has no zero in $I((1-i))$. Now F has a zero in $I((i))$ if $f((i)) \in \{0,2\}$ and if the restricted function has a zero, which gives probability $\frac{2}{3}\frac{3}{4} = \frac{1}{2}$. Thus the combined probability that $F \in Z_S$ is $\frac{1}{4}$.

Case II Suppose next that $\sigma = (2)$. Then $F \in Z_S$ if and only if F has zeroes in both $I((0))$ and $I((1))$. It follows from the argument in Case I that $\mu^* * (Z_S) = \frac{1}{4}$.

Notice that $Z_{\{\emptyset\}} = \{F : F \text{ has no zeroes}\}$ has positive measure $\frac{1}{4}$ but $\mu^*(\{\emptyset\}) = 0$.

Now suppose $|\sigma| = n$ and let $\tau = \sigma^\frown i$; suppose by induction that $\mu^{**}(Z_{I(\sigma)}) \leq \mu^*(I(\sigma))$. Interpret τ as the code for a (finite) binary tree and let $\rho \in \{0,1\}^*$ be the terminal node of that tree such that i indicates the branching of ρ. Again there are two cases.

Case I Suppose first that $i \in \{0,1\}$. Then $F \in Z_{I(\tau)}$ if and only if $F \in Z_{I(\sigma)}$ and furthermore F has a zero in $I((\rho^\frown i))$ and has no zero in $I((\rho^\frown 1 - i))$. It follows as above that $\mu^{**}(Z_{I(\tau)}) = \frac{1}{4}\mu^{**}(Z_{I(\sigma)}) = (\frac{1}{4})^{n+1}$.

Case II Suppose next that $i = 2$. Then $F \in Z_{I(\tau)}$ if and only if F has zeroes in both $I(\rho^\frown 0)$ and $I(\rho^\frown 1)$. It follows as above that $\mu^{**}(Z_{I(\tau)}) = \frac{1}{4}\mu^{**}(Z_{I(\sigma)}) = (\frac{1}{4})^{n+1}$.

An arbitrary open set is a disjoint union of intervals and thus the desired inequality can be extended to open sets. □

Theorem 3.2. *For any random continuous function $G : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, the set of zeroes of G is either empty or is a random closed set.*

Proof. Suppose that G is a random continuous function which has at least one zero, and let $S_0, S_1, \ldots$ be a Martin-Löf test in $\mathcal{C}$. Then there is a computable function ϕ such that $S_i = \cup_n I(\sigma_{\phi(i,n)})$. We may assume without loss of generality $\mu^*(S_i) \leq 2^{-i-2}$ and that each S_i is not clopen and that, for each i, the intervals $I(\sigma_{\phi(i,n)})$ are pairwise disjoint. We will define a Martin-Löf test $S'_0, S'_1, \ldots$ in the space $\mathcal{F}$ and use the fact that G must satisfy $\{S'_i\}_{i\in\omega}$ to show that $\mathcal{Z}(G)$ satisfies $\{S_i\}_{i\in\omega}$.

Fix an interval $I(\sigma)$ in $\mathcal{C}$ and let $C_\sigma = Z_{I(\sigma)}$. Observe that there is a clopen set $B_\sigma \subseteq 2^{\mathbb{N}}$ and a corresponding finite set $\tau_0, \ldots, \tau_{k-1}$ of strings such that $B_\sigma = \cup_{j<k} I(\tau_j)$, associated with σ such that, for any $Q \in \mathcal{C}$ with code r, $r \in I(\sigma)$ if and only if $Q \subseteq B_\sigma$ and $Q \cap I(\tau_j) \neq \emptyset$ for all $j < k$. It follows that C is a difference of Π^0_1 classes. That is, $F \in C$ if and only if the following two conditions hold.

(i) For each j, F has a zero in $I(\tau_j)$; by compactness, this is equivalent to saying that for any ℓ, there is an extension $\tau \in \{0,1\}^\ell$ of τ_j such that $f(\tau) \in \{0,2\}^{|\tau|}$, where f is the function on strings representing F.

(ii) F has no zeroes outside of B. Let $2^{\mathbb{N}} - B = \cup_{\tau\in A} I(\tau)$. By compactness, F has no zeroes outside of B if and only if

$$(\exists \ell)(\forall \tau \in A)(\forall \tau' \succeq \tau)\,[|\tau'| = \ell \Rightarrow (\exists m)(f(\tau' \lceil m) = 1)]\,. \tag{2}$$

Note that the measure of C_σ may be computed uniformly from σ given the calculation from Corollary 2.13 that whenever $f(\sigma) \in \{0,2\}^{|\sigma|}$, then the probability that F has a zero in $I(\sigma)$ is exactly $\frac{3}{4}$. For each σ, we will uniformly compute a c.e. open set $S_\sigma \subseteq \mathcal{F}$ such that $C_\sigma \subseteq B_\sigma$ and such that $\mu^{**}(B_\sigma) \leq 2 \cdot \mu^{**}(C_\sigma)$. There are two stages in the construction of B_σ.

Stage I: Let U be the set of codes σ' for partial functions f' such that (2) holds with f' in place of f, and such that furthermore for every j and ℓ such that f' is defined on all length-ℓ extensions τ of τ_j, there is such a τ with $f'(\rho) \in \{0, 2\}\ \forall \rho \preceq \tau$. It is clear that for any $F \in C_\sigma$, there exists $\sigma' \in U$ with $F \in I(\sigma')$ and hence

$$C_\sigma \subseteq \bigcup \{I(\sigma') : \sigma' \in U\}.$$

As usual, we may then uniformly compute from U a set U' such that the intervals $I(\sigma')$ for $\sigma' \in U'$ are pairwise disjoint in $\mathcal{F}$ and

$$\bigcup \{I(\sigma') : \sigma' \in U\} = \bigcup \{I(\sigma') : \sigma' \in U'\}.$$

For each $\sigma' \in U'$, let $Q(\sigma') \subseteq I(\sigma)$ be the Π^0_1 class in $\mathcal{F}$ consisting of those extensions of σ' which actually have zeroes in each $I(\tau_j)$. Then in fact we have

$$C_\sigma = \bigcup \{Q(\sigma') : \sigma' \in U'\}.$$

As noted above, we can actually compute the measure $\mu^{**}(Q(\sigma'))$ uniformly from σ' by expressing $Q(\sigma')$ as an effective decreasing intersection of clopen sets. Thus for each σ', we can compute a clopen set $B(\sigma')$ such that $Q(\sigma') \subseteq B(\sigma') \subseteq I(\sigma')$ and $\mu^{**}(B(\sigma') \leq 2 \cdot \mu^{**}(Q(\sigma'))$. Let

$$B_\sigma = \bigcup \{B(\sigma') : \sigma' \in U'\}.$$

Then we have $C_\sigma \subseteq B_\sigma$ and $\mu^{**}(B_\sigma) \leq \mu^{**}(C_\sigma)$.

Finally, for each i, let

$$S'_i = \cup_n B_{\sigma'_{\phi(i,n)}}.$$

Then by Proposition 2.9, $\mu^{**}(S'_i) \leq 2 \cdot \mu^*(S_i) \leq 2^{-i-1}$ and therefore there exists some i such that $G \notin S'_i$, since F is random. But this means that $\mathcal{Z}(G) \notin S_i$ and hence $\mathcal{Z}(F)$ meets the Martin-Löf test. Thus $\mathcal{Z}(F)$ is random, as desired. □

3.1 Distance functions

The space $2^{\mathbb{N}}$ has metric δ defined by

$$\delta(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 2^{-n}, & \text{if } n \text{ is the least such that } x(n) \neq y(n). \end{cases}$$

This may be viewed as a computable mapping from $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ into $2^{\mathbb{N}}$ by representing 0 as 0^ω and 2^{-n} as $0^n \frown 1 \frown 0^\omega$.

For any closed set Q in $2^{\mathbb{N}}$, the distance function d_Q may be defined as

$$\delta_Q(x) = min\{d_Q(x, y) : y \in Q\}.$$

That is,

$$\delta_Q(x) = \begin{cases} 0, & \text{if } x \in Q; \\ 2^{-n}, & \text{where } n \text{ is the least such that } x\lceil n \notin T_Q, \text{ otherwise.} \end{cases}$$

We note that the distance function of an effectively closed set is not always computable. We will say that $\delta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a *pseudo-distance* function for the set Q if Q is the set of zeroes of δ. Then it is easy to see that Q is a Π^0_1 class if and only if Q has a computable pseudo-distance function. The distance function δ_Q based on δ as defined above can never be random, since for any $\sigma \notin T_Q$, δ is constant on the interval $I(\sigma)$. If Q possesses a random pseudo-distance function δ, then it is the set of zeroes of δ and hence is a random closed set by Theorem 3.2.

We conjecture that the converse also holds, that is, any random closed set possesses a random pseudo-distance function.

4 Conclusions and Future Research

In this paper we have proposed a notion of randomness for continuous functions on the Cantor space $2^{\mathbb{N}}$ and derived several properties of random continuous functions. Random Δ^0_2 continuous functions exist, but no computable function can be random. In fact, no random function can map a computable real to a computable, or even c.e. real. We have shown that the image of a random continuous function is always a perfect set and hence uncountable. For any $y \in 2^{\mathbb{N}}$, there exists a random continuous function F with y in the image of F. Thus the image of a random continuous function need not be a random closed set. We have shown that the set of zeroes of a random continuous function is a random closed set and we conjecture that the converse is also true.

We remark that one could also define n-random closed sets and continuous functions and show that, for example, the set of zeroes of an n-random continuous function is an n-random closed set.

We would like to extend the notion of a random continuous function to functions on the real unit interval $[0, 1]$ and the real line $\Re$ by representing functions again in terms of the images of subintervals. We conjecture that a random continuous real function cannot be left or right computable and in fact, not weakly computable. We also conjecture that a random continuous function is nowhere differentiable.

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