INVARIANCE IN $\mathcal{E}^*$ AND $\mathcal{E}_\Pi$

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ABSTRACT. We define G, a substructure of $\mathcal{E}_\Pi$ (the lattice of Π^0_1 classes) and show that a quotient structure of G, $G^\diamondsuit$, is isomorphic to $\mathcal{E}^*$. The result builds on the Δ^0_3 isomorphism machinery, and allows us to transfer invariant classes from $\mathcal{E}^*$ to $\mathcal{E}_\Pi$, though not, in general, orbits. Further properties of $G^\diamondsuit$ and ramifications of the isomorphism are explored, including degrees of equivalence classes and degree invariance.

1. INTRODUCTION

A Π^0_1 class may be defined as the collection of infinite paths through a computable subtree of $2^{<\omega}$, the complete binary-branching tree. Π^0_1 classes have become a fundamental notion in computability theory because of their ability to code a wide range of constructions. For example, the collection of ideals of a computably enumerable (c.e.) commutative ring forms a Π^0_1 class. For this and other examples, as well as a survey of results about Π^0_1 classes, see [1], [3], and [5].

The lattice of all Π^0_1 classes is called $\mathcal{E}_\Pi$, by analogy with $\mathcal{E}$, the lattice of computably enumerable (c.e.) sets. The properties of $\mathcal{E}$ have been extensively studied (for a survey, see [19], chapters X and XV). Research on Π^0_1 classes and $\mathcal{E}_\Pi$ is currently quite active, with many open questions (see [3] for a number of examples). However, relatively little is known about the orbits and invariant classes of $\mathcal{E}_\Pi$. The goal of the research presented here is to expand that knowledge, in particular by transferring information to $\mathcal{E}_\Pi$ from $\mathcal{E}^*$, the lattice of c.e. sets modulo finite difference.

A Π^0_1 class P is principal (or clopen) if there is a finite set F of nodes of $2^{<\omega}$ such that an infinite path of the tree is in P if and only if it extends some $\sigma \in F$. Cholak, Coles, Downey, and Herrmann [7] showed that there were at most two non-isomorphic intervals of the form $[P, 2^\omega]$

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in $\mathcal{E}_\Pi$: those where P is principal and those where it is nonprincipal. Cenzer and Nies [4] showed that these are in fact distinct cases.

Nies proceeded to define $G = [P, 2^\omega]$ for P nonprincipal. It is via G that we will transfer information from $\mathcal{E}^*$ to $\mathcal{E}_\Pi$, and many of Nies' unpublished results are reproduced in §3-4. Several of the results are directly proved in the setting of Π^0_1 classes. However, although the goal is to transfer information to $\mathcal{E}_\Pi$, it is generally more straightforward to approach G from a different perspective, that of c.e. ideals (see §2).

Prior to being investigated in this context, G (as a collection of ideals) arose as part of the study of the lattice of c.e. substructures of a computably presented model, an area suggested by Metakides and Nerode in a 1975 paper [15]. A number of papers emerged studying substructures of particular models, such as vector spaces, algebraically closed fields, and Boolean algebras (see Nerode and Remmel [16] for references). Remmel [17] and later Downey [9, 10] generalized the work on specific structures to results about effective closure systems (M, cl), where M is a computable set and $\mathrm{cl} : \mathcal{P}(M) \to \mathcal{P}(M)$ is an effective closure operator, a map with certain properties. For example, the operator could take a subfield of M to its algebraic closure within M. As part of this work the notion of equivalence modulo finite difference, as in $\mathcal{E}^*$, is extended to equivalence modulo "finitely-generated difference." That is, $A =^* B$ if there is a finite set X such that $\mathrm{cl}(A \cup X) = \mathrm{cl}(B \cup X)$. Downey in particular gives a long list of examples of effective closure systems which includes the remark that in order to keep the lattice of c.e. ideals from collapsing under $=^*$, one must restrict the domain, and suggests fixing a maximal ideal to work within (see [9] §2 Example 8). This restriction gives the structure herein called G.

We put the same equivalence relation on G as in [9, 10, 17], where the closure operator takes a set to the ideal it generates. Since $=^*$ has been used in other work on Π^0_1 classes to mean finite difference literally, we will use $=^\diamondsuit$ for finitely-generated difference. We denote $G/=^\diamondsuit$ by $G^\diamondsuit$. Definitions and basic results for $G^\diamondsuit$ may be found in §3.

As we will see, the structure $G^\diamondsuit$ exhibits remarkable similarity to $\mathcal{E}^*$. André Nies and the author have translated several significant theorems of $\mathcal{E}^*$ to $G^\diamondsuit$, where they hold with similar proofs. Examples include the Owings splitting theorem (Theorem 4.5); the existence, for any initial segment, of sets maximal in that segment; the existence of major subsets of noncomplemented elements; and the existence of an orbit of creative sets. These translations suggested a close relationship between $G^\diamondsuit$ and $\mathcal{E}^*$, and in fact all are corollaries of our main result:

Theorem 15.1. *$G^\diamondsuit$ is isomorphic to $\mathcal{E}^*$.*

The proof is put off to the end of the paper, for the sake of clarity. It draws upon the Δ^0_3 automorphism machinery developed by Cholak, Soare, Harrington, and others (the specific format follows [12]; see also [6]), which will be fully developed in the exposition, construction, and verification in §9-15.

We will show that a class of $G^\diamondsuit$ which forms an orbit or is invariant under automorphisms gives a class of $\mathcal{E}_\Pi$ which is invariant. With the above isomorphism, then, we are able to translate an invariant class of $\mathcal{E}^*$ to one of $\mathcal{E}_\Pi$. However, orbits do not in general survive the transition. In fact, we will show that any orbit of $G^\diamondsuit$ containing a Π^0_1 class of Cantor-Bendixson rank strictly less than ω_1^{CK} does not translate to an orbit of G. Invariance and orbit transfer results are presented in §5.

Unfortunately, we may not automatically translate degree-theoretic information via the isomorphism. A collection $\mathcal{D}$ of Turing degrees forms a degree invariant class in $\mathcal{E}^*$ if there is a collection $\mathcal{C}$ of c.e. sets closed under automorphisms of $\mathcal{E}^*$, such that every set in $\mathcal{C}$ has a degree in $\mathcal{D}$ and every degree in $\mathcal{D}$ has a representative set in $\mathcal{C}$. The image of $\mathcal{C}$ under the isomorphism from $\mathcal{E}^*$ to $G^\diamondsuit$ does not necessarily correspond to the same degree collection $\mathcal{D}$, as we will discuss in §7. Degree-theoretic results and open questions may also be found in §5 and §6.

Finally, §8 holds a few notes on thin and minimal Π^0_1 classes in $G^\diamondsuit$.

2. Preliminaries

As usual, we denote the collection of computably enumerable (c.e.) sets under inclusion by $\mathcal{E}$, and the quotient structure of c.e. sets modulo finite difference by $\mathcal{E}^*$. Notation for functions and sets, and computability-theoretic terminology, will follow Soare [19].

We define a Π^0_1 class as the collection of infinite paths through a computable subtree of $2^{<\omega}$. The lattice of Π^0_1 classes ordered by inclusion is denoted $\mathcal{E}_\Pi$, after $\mathcal{E}$. For basic properties of Π^0_1 classes, see [1, 3, 5].

The countable atomless Boolean algebra is denoted Q. We view Q as a collection of propositional formulas modulo tautological equivalence, where the independent elements $\{p_i : i \in \omega\}$ generate Q. That is, letting $\epsilon_i p_i$ stand for either p_i or $\neg p_i$, a typical element of Q is a collection of logically equivalent formulas, each of which may be put into the form

$$\bigvee_{j=1}^{n} \bigwedge_{k=1}^{m} \epsilon_{jk} p_{i_{jk}}$$

for some $n, m \in \omega$.

Elements of $\{p_i, \neg p_i\}_{i\in\omega}$ are called *literals*. We order Q by logical implication. Note that while in most formulas the symbol & will be used for conjunction, within elements of Q we will use the symbol $\wedge$.

Finite strings (elements of $2^{<\omega}$) will in general be denoted by lowercase Greek letters, especially σ and τ, and infinite strings (elements of 2^ω) by lowercase Roman letters, especially f and g. The notation for elements of Q will depend on the context. The empty string in $2^{<\omega}$ is denoted λ, and the length of a string σ is $|\sigma|$. If τ extends σ, we write $\sigma \subseteq \tau$; if that extension is certainly proper, we write $\sigma \subset \tau$. The symbol $\perp$ indicates two elements which are *disjoint* or *incomparable*. In Q, $\varphi \perp \psi$ means $\varphi \not\to \psi$ and $\psi \not\to \varphi$. In $2^{<\omega}$, $\sigma \perp \tau$ means $\sigma \nsubseteq \tau$ and $\tau \nsubseteq \sigma$. A sequence is *pairwise disjoint* if each element of the sequence is disjoint from every other element. For a string f, $f \upharpoonright i$ is the *initial segment of f of length i*; that is, the unique string $\sigma \in 2^{<\omega}$ of length i such that $\sigma \subseteq f$. The concatenation of the string τ onto the end of the string σ will be denoted $\sigma^\frown\tau$.

For a string $\sigma \in 2^{<\omega}$, $[\sigma]$ is the *interval generated by* σ or *cone above* σ, which means either $\{f \in 2^\omega : \sigma \subset f\}$ or $\{\tau \in 2^{<\omega} : \sigma \subseteq \tau\}$, depending on context. Intervals are both closed and open in the topology of 2^ω and of $2^{<\omega}$, and so finite unions of intervals are also both closed and open, which will be abbreviated *clopen*. A Π^0_1 class which is clopen in the topology of 2^ω is also called *principal*.

Definition 2.1. *A subset I of Q is called an* ideal *if*

(i) $\sigma, \tau \in I \;\Rightarrow\; \sigma \vee \tau \in I$
(ii) $(\sigma \in I \;\wedge\; \tau \in Q) \;\Rightarrow\; \sigma \wedge \tau \in I$

The ideal I in the above definition is called a c.e. ideal if it is computably enumerable as a set. The ideal *generated* by a set X, denoted $\langle X\rangle$, is the closure of X under the implications above. If an ideal may be generated by a finite subset of Q (equivalently, by a single element of Q), it is called *principal*.

The collection of all c.e. ideals of Q is called $I(Q)$, and forms a lattice. The greatest element is Q, the least is 0 (the collection of logically contradictory formulas), the join of X and Y is $X \vee Y = \langle X \cup Y\rangle$, and the meet is $X \cap Y$.

Definition 2.2. *A subset I of $2^{<\omega}$ is called an* ideal *if*

(i) $\sigma^\frown 0, \sigma^\frown 1 \in I \;\Rightarrow\; \sigma \in I$
(ii) $(\sigma \in I \;\wedge\; \sigma \subseteq \tau) \;\Rightarrow\; \tau \in I$

Again, an ideal is called c.e. if it is c.e. as a set. In $2^{<\omega}$, an ideal X has a *root set*; that is, a collection of pairwise disjoint strings $\{\sigma_i\}_{i\in I}$

which generate X, with the minimality property that if $\tau \in X$ and $\tau \subseteq \sigma_i$, then $\tau = \sigma_i$. The root set is finite exactly when X is principal.

The collection of all c.e. ideals of $2^{<\omega}$ is denoted $I(2^{<\omega})$, a lattice with greatest element $2^{<\omega}$ and least element $\emptyset$.

Lemma 2.3 ([7] 2.5, equivalent form)**.** *$I(Q)$ and $I(2^{<\omega})$ are computably isomorphic in a natural way.*

Next we associate $\mathcal{E}_\Pi$ with $I(Q)$, via $I(2^{<\omega})$. Let T be a computable subtree of $2^{<\omega}$, so that $[T]$ is a Π^0_1 class. A node of $2^{<\omega}$ with no extension in $[T]$ is called a *nonextendible* node of T (note that this set includes every node in $\overline{T} = 2^{<\omega} - T$). The following claims are easily checked.

Claim 2.4. *Let T be a computable binary-branching tree. The collection of all nonextendible nodes of T forms an ideal of $2^{<\omega}$; in fact, it is equal to $\langle\overline{T}\rangle$.*

Note that if T and T' are trees such that $[T] = [T']$, then $\overline{T}$ and $\overline{T'}$ generate the same ideal of $2^{<\omega}$, by the definition of nonextendible.

Claim 2.5. *Every ideal of $2^{<\omega}$ is the set of nonextendible nodes of some Π^0_1 class.*

Thus the map $T \mapsto \langle\overline{T}\rangle$ gives a well-defined bijective correspondence between ideals of $2^{<\omega}$ and Π^0_1 classes. In fact, it is a computable isomorphism, and therefore $I(Q)$ and $\mathcal{E}_\Pi$ are computably isomorphic as well. Notice that the isomorphism is order-reversing, since a larger Π^0_1 class has fewer nonextendible nodes and thus corresponds to a smaller ideal. In particular we have the following result.

Proposition 2.6. *Under the isomorphism above, a maximal ideal of $2^{<\omega}$, and thus of Q, corresponds to a singleton Π^0_1 class.*

Corollary 2.7. *A maximal ideal of $2^{<\omega}$ has a computable root set.*

There are some technical details of ideals to cover, in order to streamline matters later on. A sequence of elements $\{a_i\}_{i\in I}$ of Q (respectively, $2^{<\omega}$) which is pairwise disjoint, as defined before, has the property that for all $i, j, \in I$, $i \neq j$, $\langle a_i\rangle \cap \langle a_j\rangle = 0$ (respectively, $\emptyset$). Note that given an arbitrary c.e. sequence $\{a_i\}_{i\in\omega}$ generating an ideal $A \in I(Q)$, one can construct a pairwise disjoint c.e. generating sequence $\{\hat{a}_i\}_{i\in\omega}$ for A. Let $\hat{a}_i = a_i \wedge \neg(\vee_{j<i} a_j)$. It is easy to see that sequence fulfills the requirements.

Now we standardize the enumeration of an ideal. Any principal ideal is computable, so we may refer to it without using an enumeration. Given a c.e. generating sequence $\{a_i\}_{i\in\omega}$ for the ideal A, define A_s as the principal ideal generated by $\{a_i : i \leq s\}$. In §9-15 the enumeration will be defined differently, but unless otherwise stated $\{A_s\}$ is a nested sequence of principal ideals.

3. Initial Definitions and Results for G

Recall that $I(Q)$ is the lattice of computably enumerable ideals of Q, the countable atomless Boolean algebra.

Theorem 3.1 ([7] 3.9, equivalent form)**.** (i) *If $I \in I(Q)$ is nontrivial and principal, then $[0, I] \cong I(Q)$.*
(ii) *If $I, J \in I(Q)$ are nonprincipal, then $[0, I] \cong [0, J]$.*
(iii) *The isomorphisms above are computable.*

Herrmann conjectured that if $I \in I(Q)$ is principal and $J \in I(Q)$ is nonprincipal, then $[0, I] \ncong [0, J]$. His conjecture was proven by Cenzer and Nies.

Theorem 3.2 ([4] 4.1, equivalent form)**.** *Let $I \in I(Q)$ be nonprincipal. Then $[0, I] \ncong I(Q)$.*

Definition 3.3 (Nies)**.** *$G = [0, M] \subset I(Q)$, an initial segment of $I(Q)$ under inclusion, for any nonprincipal ideal M.*

By the theorems preceding the definition, all copies of G are isomorphic to each other but not to $I(Q)$. To distinguish different copies of G within $I(Q)$, we will use the notation $G_M = [0, M]$.

Define an equivalence relation $=^\diamondsuit$ on G by

$$A =^\diamondsuit B \iff (\exists m \in M)[A \vee \langle m \rangle = B \vee \langle m \rangle].$$

In other words, $A =^\diamondsuit B$ when their differences are contained in a principal subideal of M.

Notation. $G/{=^\diamondsuit}$ is denoted $G^\diamondsuit$.

The structure $G^\diamondsuit$ is essentially G modulo principal ideals. Notice that $=^\diamondsuit$ depends on our choice of G.

The ordering on $G^\diamondsuit$ is set containment outside some principal ideal.

$$A^\diamondsuit \leq B^\diamondsuit \iff (\exists m \in M)[A \vee \langle m \rangle \subseteq B \vee \langle m \rangle]$$

for any $A \in A^\diamondsuit$, $B \in B^\diamondsuit$. When we are considering specific representatives A, B of $A^\diamondsuit$, $B^\diamondsuit$, we will sometimes write $A \subseteq^\diamondsuit B$ for $A^\diamondsuit \leq B^\diamondsuit$, as we might write $A^\diamondsuit = B^\diamondsuit$ for $A =^\diamondsuit B$.

André Nies presented initial results on G in a talk at the San Diego Joint Mathematics Meetings in January, 2002, and later began to consider $G^\diamondsuit$. Results attributed to Nies in this paper were stated by him in San Diego or during his visit to Notre Dame in May of 2002. Proofs have in most cases been fleshed out from sketches he provided while visiting.

Unless otherwise stated, G is to be considered as a subset of $I(Q)$. However, there are two isomorphic settings which we will work in for certain results.

First, G may be considered under duality as $[N, 2^\omega] \subset \mathcal{E}_\Pi$ for any nonprincipal Π^0_1 class N (note the order-reversal). As noted in Proposition 2.6, the case where the ideal M is maximal corresponds to N being a singleton. We may recast our previous definitions in this setting.

Here $=^\diamondsuit$ is the equivalence relation

$$P =^\diamondsuit Q \iff (\exists \text{ clopen } C)[N \subseteq C \ \wedge \ P \cap C = Q \cap C].$$

When N is a singleton $\{f\}$, $=^\diamondsuit$ simplifies to

$$P =^\diamondsuit Q \iff (\exists n)[P \cap [f \restriction n] = Q \cap [f \restriction n]].$$

In the singleton Π^0_1 class setting, $P^\diamondsuit \leq Q^\diamondsuit$ if given representatives P, Q, respectively,

$$(\exists n)[P \cap [f \restriction n] \subseteq Q \cap [f \restriction n]]. \tag{3.1}$$

The order relation for Π^0_1 classes, then, is eventual containment. Note that for P containing f, if there exists an $n \in \omega$ such that $[f \restriction n] \subseteq P$, then $P =^\diamondsuit 2^\omega$. Thus not only are the intermediate elements of $G^\diamondsuit$ nonprincipal, but indeed, they are nonprincipal in $[f \restriction n]$ for all n.

The remaining perspective we may use is that of c.e. ideals of the complete binary-branching tree, $2^{<\omega}$. The lattice $I(2^{<\omega})$ is useful because it is easy to visualize, but not every automorphism of $I(2^{<\omega})$ is induced by an automorphism of $2^{<\omega}$. We will, however, make extensive use of the $2^{<\omega}$ setting, especially G_{M_0} where $M_0 = 2^{<\omega} - \{0^n : n \in \omega\} \subset 2^{<\omega}$.

Proposition 3.4. *The order relation in G is Π^0_2 complete, and the order relation of $G^\diamondsuit$ is Σ^0_3 complete.*

Proof. We work in the Π^0_1 setting, specifically in $G = [\{0^\omega\}, 2^\omega]$. Since all copies of G are computably isomorphic, this will show the proposition for arbitrary G. Given P and Q in G, where T_P and T_Q are the corresponding computable trees, $P \subseteq Q$ if

$$(\forall \sigma \in T_P)(\exists k)(\forall |\tau| = k)[(\tau \succeq \sigma \to \tau \notin T_P) \vee \sigma \in T_Q]$$

which, since the innermost quantifier is bounded, is a Π^0_2 sentence. The sentence (3.1) defining ordering in $G^\diamondsuit$ is then Σ^0_3.

The set $\text{Tot} = \{e : W_e = \omega\}$ is Π^0_2 complete ([19] IV.3.2), and the set $\text{Cof} = \{e : W_e \text{ is cofinite}\}$ is Σ^0_3 complete ([19] IV.3.5). First we show that Tot is reducible to the ordering of G.

Given a c.e. set W_e, define a tree T_e as follows: Begin to build a complete tree. At any point that you see $n \searrow W_e$, cease extending $0^n 1$ in T_e. Then the index $e \in \mathrm{Tot}$ if and only if $[T_e] \subseteq \{0^\omega\}$.

The above construction also shows that Cof is reducible to the ordering on $G^\diamond$, because $e \in \mathrm{Cof}$ if and only if there is some n such that all $m \geq n$ are in W_e. In that case, $[T_e] \cap [0^n] = \{0^\omega\}$, so $[T_e] \subseteq^\diamond \{0^\omega\}$. □

Now we introduce examples of significant index sets for G.

Definition 3.5. *For a fixed G_M with M maximal, let W_e, $e \in \omega$, be an enumeration of all subideals of M. The following are three index sets for G_M:*

(i) Prn $= \{e : W_e$ *is principal*$\}$
(ii) Npr $= \{e : W_e$ *is nonprincipal*$\}$
(iii) Cop $= \{e : (\exists m \in M)[\overline{W}_e \subseteq \langle m\rangle]$, *that is,* W_e *is "co-principal"*$\}$

Theorem 3.6. *Prn is Σ^0_2, Npr is Π^0_2, and Cop is Σ^0_3.*

Proof. The ideal W_e is nonprincipal if every principal ideal of M omits at least one element of W_e. The index set is

$$\mathrm{Npr} = \{e : (\forall m \in M)(\exists x \in M)(\exists s)[x \in W_{e,s} \ \&\ x \notin \langle m\rangle]\},$$

which is Π^0_2 because membership in M, $\langle m\rangle$, or $W_{e,s}$ is computable. Since every ideal is principal or nonprincipal but not both, this also shows that Prn is Σ^0_2.

W_e is co-principal if its complement is contained in a principal ideal of M; that is, if e is in the set

$$\mathrm{Cop} = \{e : (\exists m \in M)(\forall x \in M)(\exists s)[x \in W_{e,s} \ \vee\ x \in \langle m\rangle]\},$$

which is a Σ^0_3 set. □

Theorem 3.7. *Npr is Π^0_2-complete, and Prn is Σ^0_2-complete.*

Proof. We will work from the $2^{<\omega}$ perspective, specifically in G_{M_0}. As in Proposition 3.4, this will show the result for all copies of G. We use the Σ^0_2-complete set Fin $= \{e : W_e$ is finite$\}$ and the Π^0_2-complete set Inf $= \{e : W_e$ is infinite$\}$ (see [19] IV.3.2).

Given a c.e. set A, let I be the ideal generated by the set $\{0^n 1 : n \in A\}$. I is a c.e. ideal in G_{M_0}. If I is nonprincipal, the given set A is infinite, and if I is principal, A is finite. Therefore Fin reduces to Prn and Inf reduces to Npr, and Prn and Npr are Σ^0_2- and Π^0_2-complete, respectively. □

4. Further comparisons between G, $G^\diamondsuit$, $\mathcal{E}$, and $\mathcal{E}^*$

Proposition 3.4 showed that G and $G^\diamondsuit$ have the same order relation complexities as $\mathcal{E}$ and $\mathcal{E}^*$, respectively. Theorem 3.7 showed that the index sets of principal and nonprincipal ideals in G correspond in complexity to the index sets of finite and infinite sets, respectively, in $\mathcal{E}$. There is another interesting connection between G and $\mathcal{E}$.

Proposition 4.1 (Nies)**.** *G contains $\mathcal{E}$ as an end segment.*

Proof. Fix $G_M \subset I(Q)$ and let $\{m_i\}$ be a disjoint list of generators for M. Define (uniformly) a sequence M_i of maximal subideals of m_i, and let $C = \bigcup_i M_i$. Then we can map $\mathcal{E}$ to $[C, M] \subseteq G$ isomorphically by $V \in \mathcal{E} \mapsto C \cup < m_i : i \in V >$. □

It should be noted that Downey proved a similar proposition for $G^\diamondsuit$, showing there is a subinterval of $G^\diamondsuit$ effectively isomorphic to $\mathcal{E}^*$ ([9] Lemma 3.1; see also the corrigendum [10]).

The next question is whether any pair of these structures are isomorphic. With G we obtain only negative results. G is not isomorphic to $\mathcal{E}$ because $\mathcal{E}$ has atoms (the singleton sets) and G does not; a nontrivial ideal always has proper subideals. G is, furthermore, not isomorphic to $\mathcal{E}^*$, because in $\mathcal{E}^*$ all nontrivial complemented elements share an orbit. In G, the principal ideal $\langle m\rangle$, for example, does not share an orbit with its complement, $M \cap \overline{\langle m\rangle}$. As will be seen in Corollary 5.5, all automorphisms of G are induced by those of M, so a principal ideal cannot map to a nonprincipal ideal.

We are left to consider possible isomorphisms involving $G^\diamondsuit$, with $\mathcal{E}$, $\mathcal{E}^*$, or G. In fact, we have the following theorem:

Theorem 15.1. *$G^\diamondsuit$ is isomorphic to $\mathcal{E}^*$.*

As a corollary, we see $G^\diamondsuit$ is not isomorphic to either $\mathcal{E}$ or G. The proof of Theorem 15.1 is quite long and has been put off to the end of this paper. The exposition and definitions for the isomorphism are found in §9-13. The construction itself is in §14, and the verification in §15. The isomorphism as constructed is Δ^0_3; it is open whether that complexity bound is tight.

Initially, we believed such an isomorphism was impossible, and so tried to construct substructures of $G^\diamondsuit$ which could not exist in $\mathcal{E}^*$, such as an end segment composed of three elements. However, we ultimately proved a translation of the Owings Splitting Theorem, Theorem 4.5 below, putting an end to our efforts to find a distinction between $G^\diamondsuit$ and $\mathcal{E}^*$. The translated Owings Splitting in $G^\diamondsuit$ is a corollary of the isomorphism between $\mathcal{E}^*$ and $G^\diamondsuit$, Theorem 15.1. A version also holds for G, Corollary 4.6 below.

The Owings Splitting Theorem states that a c.e. set that is noncomplemented in an interval may be split into two disjoint c.e. sets that are noncomplemented in the same interval. In order to translate it we must consider complementation in G and $G^\diamondsuit$. Complementation in G is standard; however, unlike in $I(Q)$ as a whole, being complemented in G is not equivalent to being principal. For example, in $2^{<\omega}$ with $M_0 = 2^{<\omega} - \{0^n : n \in \omega\}$ as before, the ideal $\langle 0^{2n}1 : n \in \omega\rangle$ is nonprincipal but complemented in G_{M_0} by $\langle 0^{2n+1}1 : n \in \omega\rangle$. Complementation in $G^\diamondsuit$ requires a definition.

Definition 4.2. *Let $C^\diamondsuit < B^\diamondsuit$ be elements of $G^\diamondsuit$. The equivalence class $\widetilde{B}^\diamondsuit$ is a complement of $B^\diamondsuit$ over $C^\diamondsuit$ if*

(1) $(\exists m \in M)[(B \cap \widetilde{B}) \vee \langle m\rangle = C \vee \langle m\rangle]$
(2) $(\exists n \in M)[B \vee \widetilde{B} \vee \langle n\rangle = M]$

For $B^\diamondsuit < I^\diamondsuit$, I a c.e. ideal, $\widetilde{B}^\diamondsuit$ is a complement of $B^\diamondsuit$ in $[C, I]^{\diamondsuit_M}$ if we replace (2) above with

(2′) $(\exists n \in M)[B \vee \widetilde{B} \vee \langle n\rangle = I \vee \langle n\rangle]$.

Unfortunately, unlike the case of $\mathcal{E}$ and $\mathcal{E}^*$, the complemented elements of G and $G^\diamondsuit$ are not the same. Of course all elements complemented in G are complemented in $G^\diamondsuit$, but the converse is not true. As an example, inside $[0, M_0]$ define

$$B = \langle 0^{2n}1, 1^n0 : n \geq 1\rangle.$$

That is, B contains all the intervals off of the path of all ones, and every other interval off the path of all zeroes. The ideal B is noncomplemented in G_{M_0} because its complement must contain $\{1^n : n \in \omega\}$. For any n, the ideal $\langle 1^n\rangle$ contains 1^n0, so every ideal containing $\{1^n : n \in \omega\}$ has nonempty intersection with B and is thus not a complement. However, in $G^\diamondsuit_{M_0}$, B is complemented by

$$\widetilde{B} = \langle 0^{2n-1}1 : n \geq 1\rangle$$

because $B \cap \widetilde{B} = \emptyset$ and $B \vee \widetilde{B} \vee \langle 1\rangle = M_0$.

Proposition 4.3. *An element of $G^\diamondsuit$ is complemented if and only if it contains a complemented element of G.*

Proof. The "if" direction is clear from the fact that a complement in G is a complement in $G^\diamondsuit$. We must show that every complemented element of $G^\diamondsuit$ contains a complemented element of G.

Let B be a c.e. ideal such that $B^\diamondsuit$ is complemented. Then there exists some $\widetilde{B}$ and $n, m \in M$ so $B \cap \widetilde{B} \subseteq \langle m \rangle$ and $B \vee \widetilde{B} \vee \langle n \rangle = M$. Let $C = \langle n \rangle \vee \langle m \rangle$ and $I = B \cap \overline{C}$. Then $I =^\diamondsuit B$ and I is complemented by $\widetilde{B} \vee C$ in G. □

We are now ready to state the Owings Splitting Theorem for G and $G^\diamondsuit$. First we recall the statement of the theorem for $\mathcal{E}$.

Theorem 4.4 (Owings Splitting)**.** *Let $C \subseteq B$ be c.e. sets such that $B - C$ is not co-c.e. (that is, B is not complemented over C). Then there exist c.e. sets A_0, A_1 such that*

(1) $A_0 \cap A_1 = \emptyset$
(2) $A_0 \cup A_1 = B$
(3) $A_i - C$ *is not co-c.e. for* $i = 0, 1$
(4) For any c.e. set W, $i = 0, 1$,
$C \cup (W - B)$ *not c.e.* $\Rightarrow C \cup (W - A_i)$ *not c.e.*

Note that the last condition states for $W \supseteq B$ that if B is not complemented in $[C, W]$, then neither is $A_i \cup C$. For a proof of the theorem, see [19] (X.2.5).

The following is the translation of Owings Splitting to $G^\diamondsuit$, a corollary of Theorem 15.1.

Theorem 4.5. *Let $C^\diamondsuit < B^\diamondsuit$ be elements of $G^\diamondsuit$ such that $B^\diamondsuit$ is noncomplemented over $C^\diamondsuit$. Then there exist c.e. ideals $A_0, A_1 \subseteq M$ such that*

(1) $(\exists m \in M)[A_0 \cap A_1 \subseteq \langle m \rangle]$
(2) $(\exists n \in M)[A_0 \vee A_1 \vee \langle n \rangle = B \vee \langle n \rangle]$
(3) $A_i^\diamondsuit \vee C$ *is noncomplemented over* $C^\diamondsuit$, $i = 0, 1$
(4) For any c.e. ideal $I \subseteq M$, if $B^\diamondsuit < I^\diamondsuit$ and $B^\diamondsuit$ is noncomplemented in $[C, I]^{\diamondsuit_M}$, then $A_i^\diamondsuit \vee C$ is also noncomplemented in $[C, I]^{\diamondsuit_M}$ for $i = 0, 1$.

Corollary 4.6. *The Owings Splitting Theorem also holds in G. That is, if $C \subseteq B$ are elements of G such that B is noncomplemented over C, there exist c.e. ideals $A_0, A_1 \subseteq M$ such that*

(1) $A_0 \cap A_1 = 0$
(2) $A_0 \vee A_1 = B$
(3) $A_i \vee C$ *is noncomplemented over* C, $i = 0, 1$
(4) For any c.e. ideal $I \subseteq M$, if $B \subseteq I$ and B is noncomplemented in $[C, I]$, then $A_i \vee C$ is also noncomplemented in $[C, I]$ for $i = 0, 1$.

Proof. Let $\hat{A}_0$ and $\hat{A}_1$ be a splitting of $B^\diamondsuit$ obtained using Theorem 4.5. Since containment and complementation are more restrictive in G than in $G^\diamondsuit$, properties (3) and (4) are already satisfied. In fact, any representatives of $\hat{A}_0^\diamondsuit$ and $\hat{A}_1^\diamondsuit$ will satisfy (3) and (4) in G. Therefore we must find representatives which are a split of B in G. Let $m \in M$ be such that $\hat{A}_0 \vee \hat{A}_1 \vee \langle m\rangle \subseteq B \vee \langle m\rangle$ and additionally $\hat{A}_0 \cap \hat{A}_1 \subseteq \langle m\rangle$. Note that $\hat{A}_i \vee \langle m\rangle \subseteq B \vee \langle m\rangle$ for $i = 0, 1$.

Let $A_0 = \hat{A}_0 \cap \overline{\langle m\rangle}$. It is immediate that $A_0 =^\diamondsuit \hat{A}_0$, $A_0 \subseteq B$, and $A_0 \cap \hat{A}_1 = 0$. Now we alter $\hat{A}_1$ so it is a complement to A_0 in B. Let $A_1 = B \cap (\hat{A}_1 \vee \langle m\rangle)$. Clearly $A_0 \cap A_1 = 0$ and $A_0 \vee A_1 = B$. We must show $A_1 =^\diamondsuit \hat{A}_1$. The witness is simply m. Note $A_1 \vee \langle m\rangle = (B \cap \hat{A}_1) \vee \langle m\rangle = (B \vee \langle m\rangle) \cap (\hat{A}_1 \vee \langle m\rangle)$. Since $\hat{A}_1 \vee \langle m\rangle \subseteq B \vee \langle m\rangle$, that last ideal is simply $\hat{A}_1 \vee \langle m\rangle$, which is clearly in $\hat{A}_1^\diamondsuit$. □

5. Transfer of Information from $\mathcal{E}^*$ to $I(Q)$

First we briefly consider the translation of formulas from $G^\diamondsuit$ to G and $I(Q)$, preserving truth. It has been performed in an *ad hoc* manner so far, but we may make the translation in two standardized steps. Let I and J stand for ideals, members of G. A formula φ in $G^\diamondsuit$ translates to φ' in G, where φ' is obtained by expanding $=^\diamondsuit$ and $\subseteq^\diamondsuit$. That is, φ' is obtained by replacing all instances of $I = J$ in φ with $(\exists m)[I \vee \langle m\rangle = J \vee \langle m\rangle]$, and replacing all instances of $I \subseteq J$ with $(\exists m)[I \vee \langle m\rangle \subseteq J \vee \langle m\rangle]$.

The formula ψ in G corresponds to $(\exists M)[M \text{ is maximal } \& \ \psi']$ in $I(Q)$, where ψ' is obtained from ψ by replacing all instances of $(\exists I)$ with $(\exists I \subseteq M)$ and all instances of $(\forall I)$ with $(\forall I \subseteq M)$, and likewise for quantification over individual elements.

Recall our overall goal is to transfer information from $\mathcal{E}^*$ to $\mathcal{E}_\Pi$, or equivalently, to $I(Q)$. The isomorphism between $\mathcal{E}^*$ and $G^\diamondsuit$ suggests a three-step process, beginning in $\mathcal{E}^*$ and traveling through $G^\diamondsuit$ and G on the way to $I(Q)$. As the information we are most interested in regards orbits and invariant classes, in this section we explore the relationships between automorphisms and invariance in the various structures under consideration.

The property of being maximal is definable in $I(Q)$, as is the property of being principal (it is equivalent to being complemented; see [7]). The following claim shows that maximality defines not only an invariant class, but an orbit; in fact, a Δ^0_1 orbit. It is easily verified from the $2^{<\omega}$ perspective, recalling that a maximal ideal has a computable root set.

Claim 5.1. *Any two maximal ideals of $I(Q)$ are computably automorphic.*

For the following results, recall $G_M = [0, M]$ specifies a particular copy of G.

Claim 5.2. *For G_M with M maximal, any automorphism of G_M extends to an automorphism of $I(Q)$ of the same Turing degree.*

Proof. Working via $2^{<\omega}$, let f be the path of 2^ω which is not in M. Let I be an ideal in $I(Q)$ and Φ an automorphism of G_M. Extend Φ to a map on $I(Q)$, Ψ, as follows.

$$\Psi(I) = \Phi(I \cap M) \vee \{I \cap \{f \restriction n : n \in \omega\}\}. \tag{5.1}$$

It is clear that this image is a c.e. ideal. After some checking, one may see Ψ is an automorphism.

The automorphism Ψ has the same Turing degree as the original Φ because the right-hand set in the join in (5.1) is computably enumerable. Note that an ideal $I \subseteq M$ has the same image under Ψ as it did under Φ. □

Theorem 5.3 ([7] 6.1, equivalent form)**.** *Every automorphism of $I(Q)$ is induced by a unique automorphism of Q.*

Corollary 5.4. *Every automorphism of G_M with M maximal extends to an automorphism of $I(Q)$ which is induced by a unique M-preserving automorphism of Q.*

Corollary 5.5. *Every automorphism of G is induced by a unique automorphism of M.*

Next we speak of orbits in the three structures. André Nies showed that an orbit in G induces an orbit in $I(Q)$. The claim follows almost immediately from the preceding results.

Claim 5.6 (Nies)**.** *For U an orbit of G, let U_M denote U's isomorphic copy in G_M. Then $EXT(U) = \bigcup\{U_M : M$ is a maximal ideal of $Q\}$ is an orbit of $I(Q)$ of the same complexity as U.*

Proof. Closure of $EXT(U)$ comes from the fact that containment in a maximal ideal is definable. Transitivity follows from Claims 5.1 and 5.2. The complexity of the orbit does not increase because the map in Claim 5.1 may be chosen to be computable. □

So far we have completed two of the three steps suggested for transferring information. The first step, $\mathcal{E}^*$ to $G^\diamondsuit$, is trivial because of the isomorphism. Claim 5.6 takes care of the third step, from G to $I(Q)$, by associating orbits in $I(Q)$ to orbits in G; the same procedure will work with invariant classes. Unfortunately between G and $G^\diamondsuit$ the transfer fails and we retain invariance but not, in general, orbits. Given $U^\diamondsuit$, an orbit or invariant class in $G^\diamondsuit$, define $U = \{A : A^\diamondsuit \in U^\diamondsuit\}$. The collection U must be invariant because any automorphism of G which takes an element in U to an element outside U will induce an automorphism of $G^\diamondsuit$ which does the same thing to $U^\diamondsuit$. However, U will not necessarily be an orbit even if $U^\diamondsuit$ was; that is, there may be ideals A and B in the collection such that no automorphism f of G takes A to B. The result draws on the idea of Cantor-Bendixson rank, and we work in the Π^0_1 class perspective.

Definition 5.7. *The* Cantor-Bendixson derivative *of a* Π^0_1 *class* P *is*

$$D(P) = P - \{f : f \text{ is isolated in } P\}.$$

We may iterate the derivative to get $D^2(P)$*,* $D^3(P)$*, etc., with* $D^\alpha(P) = \bigcap_{\beta<\alpha} D^\beta(P)$ *for limit ordinals* α*. The* Cantor-Bendixson rank *of* P *is the least ordinal* α *such that* $D^\alpha(P) = D^{\alpha+1}(P)$*. Let* $CB(P)$ *denote the Cantor-Bendixson rank of* P*.*

Definition 5.8. *The* computable ordinals *are the order types of computable well-orderings of* ω*. The least non-computable ordinal is* Church-Kleene ω_1*, or* ω_1^{CK}*.*

Theorem 5.9 (Kreisel [14], see [2])**.** *The set of Cantor-Bendixson ranks of* Π^0_1 *classes,* $\{\alpha : (\exists P)[CB(P) = \alpha]\}$*, is exactly the set of ordinals* $\{\alpha : \alpha \le \omega_1^{CK}\}$*.*

Theorem 5.10. *Let* $A^\diamondsuit \in G^\diamondsuit$*. The set* $\{CB(P) : P \in A^\diamondsuit\}$ *is closed upwards in the ordinals* $\le \omega_1^{CK}$*.*

Proof. Let $A \in A^\diamondsuit$ such that $CB(A) = \alpha < \omega_1^{CK}$, and let $\beta > \alpha$ be a computable ordinal or ω_1^{CK}. Supposing $G = [N, 2^\omega]$, let $p \in 2^{<\omega}$ such that $[p] \cap N = \emptyset$. From Theorem 5.9, let P be a Π^0_1 class of rank β. Let $Q = (A \cap \overline{[p]}) \cup \{p^\frown f : f \in P\}$ is a Π^0_1 class in $A^\diamondsuit$ of rank β. □

The following theorem is well-known.

Theorem 5.11. *Cantor-Bendixson rank is* $\mathcal{L}_{\omega_1\omega}$*-definable in the language of inclusion, so preserved under automorphisms of* $\mathcal{E}_\Pi$ *or* G*.*

Corollary 5.12. *For any equivalence class* $A^\diamondsuit$ *in* $G^\diamondsuit$ *which contains a* Π^0_1 *class of Cantor-Bendixson rank* $< \omega_1^{CK}$*, there are classes* $A, B \in A^\diamondsuit$ *which are not automorphic in* G*.*

Proof. By Theorems 5.9 and 5.10, $X = \{CB(P) : P \in A^\diamondsuit\}$ is a subset of the ordinals $\leq \omega_1^{CK}$ which is closed upward. Therefore for any computable ordinal $\alpha \in X$ there is an ordinal $\beta > \alpha$ such that $\beta \in X$ also. Thus as long as there is some element of $A^\diamondsuit$ with computable ordinal rank, there exist elements A, B of $A^\diamondsuit$ with different Cantor-Bendixson rank. Since automorphisms of G must preserve Cantor-Bendixson rank, A and B cannot be automorphic in G. □

Corollary 5.13. *Any orbit* $\mathrm{Orb}(A^\diamondsuit)$ *of* $G^\diamondsuit$ *generated by a* Π^0_1 *class* A *such that* $CB(A) < \omega_1^{CK}$ *corresponds to an invariant class in G which is not an orbit.*

Corollary 5.13 leaves open the possibility of an orbit of Π^0_1 classes which are all of rank ω_1^{CK}. Let $\mathscr{C}$ be the collection of all ideals A which satisfy the following formula, where quantifiers range over G.

$$(\exists C \supset A)(\forall B \subseteq C)(\exists R)[R \text{ complemented } \& \ R \cap B = R \cap A \\ \& \ (\forall X =^\diamondsuit R \cap C)[X \text{ noncomplemented}]] \tag{5.2}$$

André Nies obtained (5.2) by direct translation of Harrington's definition of creativity in $\mathcal{E}$ to G (see [19] XV.1.1). He has announced the following theorem:

Theorem 5.14 (Nies)**.** *The collection of ideals* $\mathscr{C}$ *is nonempty and forms an effective orbit in* G*.*

$\mathscr{C}$ is the same collection of ideals as that obtained by pushing Harrington's definition from $\mathcal{E}$ to G via $\mathcal{E}^*$ and $G^\diamondsuit$, though the latter process gives a seemingly weaker condition than (5.2). Thus we obtain the result that all "creative ideals" must have Cantor-Bendixson rank ω_1^{CK}, and have an example of an orbit which remains an orbit when translated from $\mathcal{E}^*$ to $I(Q)$.

6. Degrees of Ideals and $\diamondsuit$-Equivalence Classes

The correspondence between ideals of $2^{<\omega}$ and ideals of Q preserves degree, so when we show facts about degrees we may use either setting, as convenient. Recall the notation that $\boldsymbol{A}$ is the Turing degree of A.

There is an ideal of every Turing degree. For the set W, let I_W be the ideal of $2^{<\omega}$ generated by the set

$$\{0^{n+1}1 : n \in W\}.$$

I_W is clearly computable from W, and from I_W we can compute its root set, which gives W. Thus $I_W \equiv_T W$.

Theorem 6.1. *The set* $\{\boldsymbol{A} : A \in A^\diamondsuit\}$ *is closed upward in the c.e. degrees.*

Proof. Without loss of generality, we work in $M_0 \subset 2^{<\omega}$. Given $A \subseteq M_0$ and a c.e. degree $\boldsymbol{B} > \boldsymbol{A}$, choose a representative ideal B of degree $\boldsymbol{B}$. Let $\widetilde{A}$ be A with the interval [01] replaced by a copy of B; that is, let $01^\frown\tau \in \widetilde{A}$ iff $\tau \in B$. It is clear that $\widetilde{A} =^\diamondsuit A$ and $\widetilde{\boldsymbol{A}} \geq \boldsymbol{B}$. In fact, since B computes both A and B, $\widetilde{\boldsymbol{A}} = \boldsymbol{B}$. □

Definition 6.2. $\boldsymbol{A}^\diamondsuit = \min\{\boldsymbol{A} : A \in A^\diamondsuit\}$*, if this minimum exists.*

This prompts the question of whether degree is a well-defined concept in $G^\diamondsuit$, or if there are equivalence classes for which the minimum does not exist. Definition 6.2 is analogous to the definition of degree of an isomorphism class of models in computable model theory (for a survey, see Knight [13]), and there are certainly isomorphism classes of models without degree. The same is true here.

To construct an equivalence class in $G^\diamondsuit$ with no degree we build an equivalence class containing an infinite descending sequence of degrees. A similar idea was used by Richter in her thesis (see [18]), where among other results she constructed a theory with no computable models, but with models whose degrees form a minimal pair. The isomorphism class of models of such a theory has no degree.

Let A be a c.e. ideal of $2^{<\omega}$. Each $B \in A^\diamondsuit$ will be equal to A except in a finitely generated ideal I. It is clear in $I(2^{<\omega})$ that we may choose I such that $A - I$ is also an ideal. Any difference between the degree of B and that of A depends on the degree of $B \cap I$. Since $A - I$ and $B \cap I$ are disjoint, $B \equiv_T (A - I) \sqcup (B \cap I) \geq_T A - I$. The degree of a member of $A^\diamondsuit$, therefore, is at lowest the degree of $A - I$ for some finitely generated ideal I.

Recall that $M_0 \subset I(2^{<\omega})$ is the ideal $2^{<\omega} - \{0^n : n \in \omega\}$. To build an equivalence class with no degree, in $G = [\emptyset, M_0]$ we will build A such that the degrees of $\{A \cap [0^n1]\}_{n\in\omega}$ form an infinite descending sequence. First, we argue that the corresponding $A^\diamondsuit$ has no degree.

Claim 6.3. *For any ideal A in the G specified above, if the degrees of $\{A \cap [0^n1]\}_{n\in\omega}$ form an infinite descending sequence in the c.e. Turing degrees, the equivalence class $A^\diamondsuit$ in $G^\diamondsuit$ has no degree. That is, the set $\{\boldsymbol{B} : B =^\diamondsuit A\}$ has no minimum element.*

Proof. Suppose the ideal $B =^\diamondsuit A$ is of minimal degree in $A^\diamondsuit$. By definition of $\diamondsuit$-equivalence, for some n, $B \cap [0^n] = A \cap [0^n]$. Therefore, as discussed above, $B \geq_T A \cap [0^n]$. However, by the condition on A, the degree of $A \cap [0^n]$ is the degree of $A \cap [0^n1]$, which is strictly greater than the degree of $A \cap [0^{n+1}1] \equiv_T A \cap [0^{n+1}]$. Therefore, the element $A \cap [0^{n+1}] \in A^\diamondsuit$ has degree strictly less than that of B, which is a contradiction. □

To build such an A, we use the Sacks Splitting Theorem to construct a uniformly c.e. sequence of c.e. sets of descending Turing degree. First recall the original theorem.

Theorem 6.4 (Sacks Splitting Theorem). *Let B and C be c.e. sets such that C is noncomputable. Then there exist low c.e. sets A_0 and A_1 such that:*

(i) *$A_0 \cup A_1 = B$ and $A_0 \cap A_1 = \emptyset$, and*
(ii) *$C \not\leq_T A_1$, for $i = 0, 1$.*

See Soare [19], VII.3.2, for the proof.

The construction of A_0, A_1 in Theorem 6.4 is effective, so there is a function $f : \omega \times \omega \to \omega$ such that for $B = W_e$ and $C = W_j$, $A_0 = W_{f(e,j)}$. We may iterate that function to obtain the following theorem.

Theorem 6.5. *Let B be a noncomputable c.e. set. Then there exists a uniformly c.e. sequence of c.e. sets $\{B_i\}_{i\in\omega}$ such that:*

(i) *$B_0 = B$,*
(ii) *$B_{i+1} <_T B_i$, for all i.*

Proof. By Corollary VII.3.4 in [19], if C in Theorem 6.4 is set equal to B, then $\emptyset <_T A_i <_T B$ for $i = 0, 1$. Therefore, in each splitting we will let the set to be split play the role of both B and C in the original theorem.

Let the function f be as defined above, taking a pair of indices to an index for a splitting. Define the functions g_i inductively, letting $g_1(e, j) = f(e, j)$ and $g_{i+1}(e, j) = f(g_i(e, j), g_i(e, j))$ for $i > 0$. Note that $g_1(e, e)$ produces the index of a set B_1, which is a split of $B = W_e$ such that $\emptyset <_T B_1 <_T B$. The index produced by $g_2(e, e)$ will be for a split of B_1 which is properly Turing-below B_1, and so on.

Suppose the original set B is given by W_e. Then the desired sequence is $B_0 = B$, $B_i = W_{g_i(e,e)}$ for $i > 0$. □

Corollary 6.6. *There exists a $\diamondsuit$-equivalence class with no Turing degree.*

Proof. Let $\{B_i\}_{i\in\omega}$ be as in Theorem 6.5. Let the ideal $A \in G$ be generated by the set $\{0^i 1^j 0 : j \in B_i\}$. Then $A \cap [0^i 1] \equiv_T B_i$, and A meets the condition in Claim 6.3, so $A^\diamondsuit$ has no degree. □

The *jump degree* of an equivalence class $A^\diamondsuit \in G^\diamondsuit$ is the minimum degree in $\{\boldsymbol{A}' : A \in A^\diamondsuit\}$. Having degree implies having jump degree, but the existence of a $\diamondsuit$-equivalence class with no degree leaves open the following question.

Question 6.7. *Is jump degree a well-defined concept in $G^\diamondsuit$?*

7. Degree Invariant Classes and Translation

We turn now to degree-theoretic concerns of G and $G^\diamondsuit$. For our purposes there are two kinds of invariance, *set invariance* and *degree invariance*, the latter defined below. Set invariance is the only kind of invariance we have been discussing thus far, where a collection of c.e. sets is invariant if it is closed under automorphisms of $\mathcal{E}$. We will also use the term "set invariant" for collections of ideals closed under automorphisms of G or $I(Q)$.

Definition 7.1. *A collection of degrees $\mathcal{C}$ is* invariant *in $\mathcal{E}$ if there is a collection of sets $\mathcal{S}$ such that*

(i) *For every degree $\boldsymbol{d} \in \mathcal{C}$, there is a set $X \in \mathcal{S}$ of degree $\boldsymbol{d}$,*
(ii) *If $X \in \mathcal{S}$ has degree $\boldsymbol{d}$, then $\boldsymbol{d} \in \mathcal{C}$, and*
(iii) *$\mathcal{S}$ is closed under automorphisms of $\mathcal{E}$ ($\mathcal{S}$ is set invariant).*

To define degree invariance in G, everywhere in the definition above replace $\mathcal{E}$ with G and "set" with "ideal."

We would like to obtain degree invariance results for $I(Q)$ using the tools of G and $G^\diamondsuit$. If we could prove invariance of $\mathcal{C}$ in $G^\diamondsuit$, we would have it for G also and thus for $I(Q)$. The naive approach is to push the corresponding invariant collection of sets, $\mathcal{S}$, from $\mathcal{E}^*$ to $G^\diamondsuit$ and consider its degree structure. However, this is not viable. Corollary 6.6 showed that Turing degree is not a well-defined concept in $G^\diamondsuit$. However, even if it were, the isomorphism construction would not guarantee that a c.e. set W had the same degree as the equivalence class of W's image.

The alternative tactic is to work directly in G. One approach is to begin with a degree-invariant class in $\mathcal{E}$ where the corresponding class of sets $\mathcal{S}$ is neatly definable, such as the non-low$_2$ degrees and the atomless sets (see [19] XI.4, XI.5). Using concepts from $G^\diamondsuit$, we may translate the definition of $\mathcal{S}$ to G and attempt to re-prove the appropriate theorems. In that approach, each degree invariant class must be translated individually. However, Cholak and Harrington [8] have proved a more general result.

Theorem 7.2 ([8] 8.5)**.** *Let*

$$\mathcal{C} = \{\boldsymbol{a} : \boldsymbol{a} \text{ is the Turing degree of a } \Sigma^0_3 \text{ set } J \geq_T \boldsymbol{0}''\}.$$

Let $\mathcal{D} \subseteq \mathcal{C}$ such that $\mathcal{D}$ is upward closed in $\mathcal{C}$. Then there is an $\mathcal{L}(A)$ property $\varphi_D(A)$ such that

$$(\forall \text{ c.e. } F)[\boldsymbol{F}'' \in \mathcal{D} \Leftrightarrow (\exists A)[\varphi_D(A) \text{ and } A \equiv_T F]].$$

As a corollary this shows degree invariance of the non-low$_n$ and high$_n$ degrees for $n \geq 2$. We have the following conjecture.

Conjecture 7.3. *The non-low$_n$ and high$_n$ classes of degrees, $n \geq 2$, may be shown invariant in G by use of the proper translation of Theorem 7.2.*

A translation of the double jump definability result would leave very few open questions about degree invariance, namely the following.

Question 7.4. *Are the high degrees invariant in G?*

The high degrees were shown to correspond to the maximal sets by Martin ([19] XI.1.5, 2.3). Maximality is definable, so the maximal sets form an invariant class, and thus the high degrees are invariant in $\mathcal{E}$.

Question 7.5. *Is Turing-completeness invariant in G?*

In $\mathcal{E}$, the creative sets are the 1-complete sets, and 1-completeness implies Turing-completeness. Harrington's lattice-theoretic definition of the creative sets ([19] XV.1.1) shows that the creative sets are an invariant class, and so form an orbit of Turing-complete sets, answering the $\mathcal{E}$ version of Question 7.5 affirmatively.

8. Transfer of Information from $\mathcal{E}_\Pi$ to $\mathcal{E}^*$

Having found a way to move information from $\mathcal{E}^*$ to $\mathcal{E}_\Pi$, the next question is whether we can work in the opposite direction. In particular, the *array non-computable degrees*, introduced by Downey, Jockusch, and Stob [11], are an invariant class in $\mathcal{E}_\Pi$, as shown in Cholak, Coles, Downey, and Herrmann [7]. Is there a "reverse translation" by which we may show they are invariant in $\mathcal{E}$? In $\mathcal{E}_\Pi$, the invariance is shown via perfect thin classes, and many other interesting results in $\mathcal{E}_\Pi$ also involve thin and minimal classes.

Definition 8.1. *An infinite Π^0_1 class P is* thin *if every subclass of P is relatively clopen in P. That is, for any $Q \subseteq P$, there is some principal Π^0_1 class C such that $Q = P \cap C$.*

Definition 8.2. *An infinite Π^0_1 class P is* minimal *if every subclass of P is finite or cofinite in P.*

A minimal class may be visualized as a tree with exactly one non-isolated path, off of which an infinite number of isolated paths branch. Notice that minimal classes are also thin, so results proved assuming thinness hold for minimal classes. The following proposition says that every thin member of G is trivial.

Proposition 8.3. *Suppose $G = [N, 2^\omega]$ contains a thin Π^0_1 class P. Then $P =^\diamond N$.*

Proof. Since $P \in G$, $N \subseteq P$. Therefore N is relatively clopen in P; that is, there is some principal C such that $N = P \cap C$. But then $P = (P \cap C) \cup (P \cap \overline{C}) = N \cup (P \cap \overline{C})$. The complement of C is also principal, so $N \cup (P \cap \overline{C}) =^\diamond N$ and $P =^\diamond N$. □

A *perfect thin* class is a thin class where every extendible node has at least two infinite paths through it. A perfect tree may be visualized as the complete tree, $2^{<\omega}$, after it has been "stretched out" to possibly add more nodes in between branching points. By Proposition 8.3, the perfect thin classes are at best trivial members of any copy of G. In fact they cannot be members of any $G \subset \mathcal{E}_\Pi$ with singleton least element, because in a thin class a computable path must be isolated, and there are no isolated paths in a perfect class. The results we use to move from G to $\mathcal{E}_\Pi$ consider only copies of G which are maximal (with singleton least element, from the $\mathcal{E}_\Pi$ perspective), and so new techniques will have to be developed to prove degree invariance of the array non-computable degrees in $\mathcal{E}$.

9. An isomorphism between $G^\diamond$ and $\mathcal{E}^*$

Here begins the promised proof of the statement $G^\diamond \cong \mathcal{E}^*$, which is the content of the remainder of the paper. The proof builds on the Δ^0_3 automorphism machinery as developed by Cholak, Soare, Harrington, and others ([6], [12]). I have followed the layout and notation in [12] closely, and the construction and verification are laid out nearly identically. This segment is designed to be self-contained, so some definitions are repeated from §2 and §3.

A summary for those already familiar with the Δ^0_3 automorphism method follows in §9.1 and §9.2. The definitions and exposition are in §10-13. The construction itself is in §14, and the verification in §15.

9.1. **Summary: definitions and basic changes.**

This subsection and the next are directed at the reader who is familiar with the Δ^0_3 automorphism method and whose primary interest is in where the isomorphism method differs. We refer specifically to Harrington and Soare [12]; all references to the "original" construction are to that paper.

Denote the countable atomless Boolean algebra by Q, and the lattice of c.e. ideals of Q by $I(Q)$. The structure G is $[0, M]$ for any nonprincipal c.e. ideal $M \subset Q$, an initial segment of $I(Q)$. Define the equivalence relation $=^\diamond$ on G by

$$A =^\diamond B \iff (\exists m \in M)[A \vee \langle m \rangle = B \vee \langle m \rangle].$$

The quotient structure $G/{=^\diamondsuit}$ is denoted $G^\diamondsuit$. For this construction, we will fix a copy of G with M maximal.

We replace ω with M, letting $\widehat{\omega}$ be as before. Player RED builds an enumeration of c.e. ideals, $\{U_n\}_{n\in\omega}$ and one of c.e. sets, $\{V_n\}_{n\in\omega}$. Player BLUE builds sets $\{\widehat{U}_n\}_{n\in\omega}$ and ideals $\{\widehat{V}_n\}_{n\in\omega}$. State is defined as before, where enumeration of ideals is as follows.

Ideal enumeration. Suppose we have already determined J_s for J an ideal. During stage $s+1$, we may enumerate some finite collection of elements of M into J_s; call that collection X. At the end of stage $s+1$ we will close the ideal J with respect to $Y_{\lambda,s+1}$, the set of all elements on the tree. That is, we let $J_{s+1} = \langle J_s \cup X\rangle \bigcap Y_{\lambda,s+1}$. Since there are only a finite number of elements on the tree at any stage, J_{s+1} will be finite for every $s \in \omega$. In this construction, ideal closure is always effective, because membership in a principal ideal is computable.

Principal versus nonprincipal. Every nontrivial ideal is infinite, so here we are concerned with the distinction between principal ideals and nonprincipal ideals. Thus, we must amend our concept of "almost every" $x \in M$. Instead of saying almost every $x \in M$ have a property φ if the set $\{x : \neg\varphi(x)\}$ is finite, we require $\{x : \neg\varphi(x)\}$ be contained in a principal ideal.

$$(\text{a.e. } x)[\varphi(x)] \iff (\exists m \in M)(\forall x \in M)[\neg\varphi(x) \Rightarrow x \in \langle m\rangle].$$

Likewise, we must replace "there exist infinitely-many x" ($\exists^\infty x$) with something more discerning. We say "there exists a nonprincipal collection of x" ($\exists^{np} x$) as shorthand for $(\forall m \in M)(\exists x \notin \langle m\rangle)$. That is, there is an element outside every principal ideal of M. Membership in a maximal ideal or a principal ideal is computable, so $\exists^{np} x$ is of the same complexity as $\exists^\infty x$.

Complexity of sets of states. For a state to be well-visited with respect to ideals means there is a nonprincipal collection of elements which have that state at some time during the construction. Almost every element leaves a non-well-resided state by the end of the construction, in the new sense of "almost every." The only changes from the original definitions are from $\exists^\infty x$ to $\exists^{np} x$ in each case. The fact that $\exists^\infty x$ and $\exists^{np} x$ have the same complexity means the properties of a state being well-visited (§12.1) and non-well-resided (§12.3) are still Π^0_2 and Σ^0_3, respectively.

Restrictions on the movement of x. In the original construction, the size of a number $n \in \omega$ is used for several restrictions on n's movement and enumeration. In this construction, we use two replacements for size. When all that is needed is a linear order on the elements of M, we use a fixed enumeration, essentially letting the "size" of x be the stage at which it is enumerated into M. When a stronger restriction is needed, we require x be outside the principal ideal generated by some initial segment of the enumeration of M. As in the original construction, we manipulate only a finite subset of M at any given stage of the construction. We are able to make the stronger restriction for two reasons: first, M is nonprincipal, so there is always an element of M independent of any given finite initial segment, and second, membership in a principal ideal is computable, so we can identify such an independent element.

9.2. **Summary: specific alterations.** The chief point at which the stronger replacement for size is used is in defining k_β^+, the bound on the set of elements which have non-well-visited α-states for $\alpha^- = \beta$ (Equation (12.7)). It is still a number, but now instead of requiring $x > k_\beta^+$ in Steps 1 and 2, we require x be outside the principal ideal generated by the first k_β^+ elements of M enumerated. This allows (and in fact requires) the pockets of nodes $\alpha \subset f$ to contain a principal collection of elements rather than a finite collection. Pockets of nodes to the left of the true path still contain finitely-many elements, and pockets to the right are emptied every time the final step (here, Step 6) is applied.

All mention of α-witnesses has been removed, as it is not necessary that the isomorphism have any special properties. Correspondingly, we do not split S_α, R_α, and Y_α into S_α^0, S_α^1, and so on. Besides that, the only change to Step 2 (moving elements down one level) is the change in k_β^+ above. Step 1 (prompt pulling from the right to ensure $\mathcal{M}_\alpha \subseteq \mathcal{E}_\alpha$) has the additional restriction that the chosen x is independent of the elements we have already seen in R_α; that is, $x \notin \langle Y_{\alpha,s} \rangle$. Steps 3, 4, 5, and 6 (formerly 11) are unchanged.

Independence considerations must also be added to Lemmas 5.1 and 5.5 of the original construction. Lemma 5.1 (now Lemma 15.2) lists the ways elements may move on the tree and be enumerated into sets; in order to retain the usefulness of the lemma, we must restrict to enumerations such that the element is independent from what was already in the ideal. Lemma 5.5 (now Lemma 15.6) cannot assert each element is enumerated into only finitely many ideals, because Step 6 will enumerate any given element into an infinite number of ideals, so the enumerations Lemma 15.6 considers are restricted in the same way as in Lemma 15.2. This change does not hamper the use of the lemma, which is in asserting Steps 1-5 and $\hat{1}$-$\hat{5}$ act finitely often between applications of Step 6. The only other lemma change is in Lemma 5.8 (here, Lemma 15.9), which now shows that for $\alpha \subset f$, $R_{\alpha,\infty} =^\diamond Y_\lambda =^\diamond M$.

10. Framework

Any terminology and notation not explicitly defined here may be found in §11. Given two enumerations, $\{U_n\}_{n\in\omega}$ of ideals and $\{V_n\}_{n\in\omega}$ of sets, we build two enumerations, $\{\widehat{U}_n\}_{n\in\omega}$ of sets and $\{\widehat{V}_n\}_{n\in\omega}$ of ideals. The $\widehat{U}_n$ are intended as images for the U_n, and the $\widehat{V}_n$ are intended as preimages for the V_n. We think of the correspondence in terms of states, where a state ν is a collection of indices such that, for x with state ν, x is in a set or ideal if and only if the index for that set or ideal is in ν. The exact definition is as follows.

Definition 10.1. *Let $\{X_n\}_{n\in\omega}$ and $\{Y_n\}_{n\in\omega}$ be two sequences of c.e. sets or ideals. The* final e-state *of x with respect to (w.r.t.) $\{X_n\}_{n\in\omega}$ and $\{Y_n\}_{n\in\omega}$ is $\nu(e,x) = \langle e, \sigma(e,x), \tau(e,x)\rangle$, where*

$$\sigma(e,x) = \{i : i \leq e \;\&\; x \in X_i\}, \text{ and}$$

$$\tau(e,x) = \{i : i \leq e \;\&\; x \in Y_i\}.$$

If a correspondence of ideals $(U_n,\, \widehat{V}_n)$ and sets $(\widehat{U}_n,\, V_n)$ is to be an isomorphism, it must certainly satisfy the following condition.

$$\begin{aligned} &(\forall \nu)(\exists^{np} x \in M)[\nu(e,x) = \nu \text{ w.r.t. } \{U_n\}_{n\in\omega} \text{ and } \{\widehat{V}_n\}_{n\in\omega}] \\ &\iff (\exists^\infty \hat{x} \in \omega)[\nu(e,\hat{x}) = \nu \text{ w.r.t. } \{\widehat{U}_n\}_{n\in\omega} \text{ and } \{V_n\}_{n\in\omega}]. \end{aligned} \tag{10.1}$$

That is, the state corresponds to a nonprincipal ideal in G if and only if the state corresponds to an infinite set in $\mathcal{E}$.

We would like, then, to talk about the *well-resided* states. A state is well-resided on the M side if the collection of elements which have that state at the end of the construction is not contained in any principal ideal (on the ω side, the collection must be infinite). However, we have the limitation that our construction be Δ^0_3, while being well-resided is Π^0_3. This necessitates worrying about the states *as elements have them* during the construction, so we split the definition into two:

Definition 10.2. *A state ν is* well-visited *on the M side if the collection of elements which have state ν during the construction is not contained in any principal ideal. On the ω side, ν is well-visited if the collection of elements which have state ν during the construction is infinite.*

Definition 10.3. *A state ν is* non-well-resided *on the M side (ω side) if it is well-visited, but at the end of the construction, the collection of elements with state ν is contained in a principal ideal (is finite).*

Well-visited is a Π^0_2 property (see the definition of $\mathcal{F}_\alpha$ in §12.1). Non-well-resided is the complement of well-resided inside the set of well-visited states. It does not immediately appear to be an improvement over well-resided, since it is still Σ^0_3 (see the definition of $\mathcal{N}_\alpha$ in §12.3), but we may approximate it with Π^0_2 predicates which essentially say "after this (fixed) value, nothing which enters the state stays."

Since states are disjoint, all we need to know to have an automorphism is that the well-resided states coincide on the M side and the ω side (Requirement (10.1)), which we will accomplish by ensuring the well-visited states and the non-well-resided states coincide.

11. Initial Definitions

11.1. **Enumerations, ideals, and the construction tree.** Fix a maximal ideal $M \subset Q$. We map from M to ω, but for clarity we rename the image $\widehat{\omega}$. Designate elements of M by lowercase Roman letters (x, y, …), and natural numbers by hatted lowercase Roman letters ($\hat{x}$, $\hat{y}$, …). On the M side we have two indexings of the computably enumerable subideals of M, $\{U_n\}_{n\in\omega}$ and $\{\widehat{V}_n\}_{n\in\omega}$. On the $\widehat{\omega}$ side we likewise have two indexings of the c.e. sets, $\{\widehat{U}_n\}_{n\in\omega}$ and $\{V_n\}_{n\in\omega}$. The enumerations $\{U_n\}$ and $\{V_n\}$ are given; the enumerations $\{\widehat{U}_n\}$ and $\{\widehat{V}_n\}$ are built in response as images and preimages, respectively. Note that the hats on the V ideals and sets are reversed with respect to which side they live in; this is the only place where such reversal takes place. We view the construction as a game between two players. Player 1 (RED) controls the U ideals and V sets, and Player 2 (BLUE) controls the $\widehat{U}$ ideals and $\widehat{V}$ sets.

The notation for ideals will be as follows. We fix an enumeration $\{m_0, m_1, \ldots\}$ of M to use throughout the construction. Given that fixed enumeration, let $x \lhd y$ indicate x is enumerated before y. Let $P_{\unlhd x}$ be the principal ideal generated by all elements of M enumerated up to and including x, and $P_{\lhd x}$ the ideal generated by all elements of M enumerated up to but not including x. When we know which $m_i \in M$ we are working with, we have the shorthand $P_{<i} := P_{\lhd m_i}$ and $P_{\leq i} := P_{\unlhd m_i}$. For $X \subseteq M$, we will also use the notation $\langle X\rangle$ to mean the ideal generated by X. When X is a finite set $\{x_1, x_2, \ldots, x_n\}$ listed explicitly we may omit the curly braces and say $\langle x_1, x_2, \ldots, x_n\rangle$.

In the construction we will use a slightly different definition of ideal enumeration than in §2. Previously for an ideal J, at stage $s+1$ we enumerated an additional element x into J_s and let J_{s+1} be the principal ideal $\langle J_s \cup \{x\}\rangle$. Here we consider only a finite initial segment $Y_{\lambda,s}$ of

M at each stage s of the construction. During stage $s+1$, we may enumerate some finite collection $X \subset M$ into J_s. At the end of stage $s+1$ we will close the ideal J with respect to $Y_{\lambda,s+1}$. That is, we let $J_{s+1} = \langle J_s \cup X \rangle \bigcap Y_{\lambda,s+1}$, which will be finite for every $s \in \omega$.

By analogy with $(\exists^\infty x) \equiv (\forall n)(\exists x > n)$ for ω, we define $(\exists^{np} x)[\varphi(x)]$ as $(\forall m \in M)(\exists x \in M)[x \notin \langle m \rangle \ \& \ \varphi(x)]$. Since M is maximal, and membership in a maximal or principal ideal is computable, there is no complexity increase over $(\exists^\infty x)$. Verbally this will be described as a *nonprincipal collection*; a set which may not itself be an ideal, but which cannot be contained in any principal ideal. Likewise, if "almost every" (a.e.) $x \in M$ has a property φ, it means that the collection of x which do not have φ is contained in a principal ideal. Recall that for A and B, two subideals of M, $A =^\diamondsuit B$ if there is some $m \in M$ such that $A \vee m = B \vee m$. We will extend that notion to situations where A and B are not necessarily ideals but simply sets of elements, so, for example, $A =^\diamondsuit \emptyset$ means A is contained in a principal subideal of M. We will abuse terminology to refer to such a set A as "principal," and to a nonprincipal collection A as simply "nonprincipal."

The construction takes place on a tree T, which we think of as a subset of ω^ω, using coding. The tree T grows downward with its root, λ, at the top. Each node α of T will control part of the construction. For example, it may build a pair U_α, $\widehat{U}_\alpha$, where for some n_α determined by the length of α, U_α is intended as an approximation to U_{n_α} and $\widehat{U}_\alpha$ as its image $\widehat{U}_{n_\alpha}$. Likewise, some nodes control V, $\widehat{V}$ pairs, and some perform other tasks; see §11.3. T will be computable, and will have a *true path* f. If the above node α is on the true path, then $U_\alpha =^\diamondsuit U_{n_\alpha}$ and $\widehat{U}_\alpha$ is the correct candidate for $\widehat{U}_{n_\alpha}$. In this construction f is not in general computable but instead is $\emptyset''$-computable, which means the sequences of images and preimages will have only a $\emptyset''$-computable (that is, Δ^0_3) presentation. The definitions of f and T are in §13.

We use the notation for trees found in [19]. The set of all infinite paths through T is denoted $[T]$. Let nodes on the tree be designated by lowercase Greek letters (α, β, γ, δ, $\ldots$), where $\beta \subseteq \alpha$ ($\beta \subset \alpha$) indicates α extends (properly extends) β. When neither $\alpha \subseteq \beta$ nor $\beta \subseteq \alpha$ is true, we write $\alpha \perp \beta$. For two strings α and β, whether they are finite or infinite, $\alpha \cap \beta$ denotes the longest string which is a substring of both α and β. Let λ denote the empty string. Let $|\alpha|$ denote the length of α, and α^- be the immediate predecessor of α if $\alpha \neq \lambda$. Let $\alpha^\frown\beta$ denote the string formed by concatenating β to the end of α. When β is the string composed of only one element b, we may write $\alpha^\frown b$ for $\alpha^\frown\beta$.

Definition 11.1. *Let $\alpha, \beta \in T$.*

(i) *For $\alpha \perp \beta$, α is to the left of β ($\alpha <_L \beta$) if*

$$(\exists a, b \in \omega)(\exists \gamma \in T)[\gamma^\frown a \subseteq \alpha \,\&\, \gamma^\frown b \subseteq \beta \,\&\, a < b].$$

(ii) *$\alpha \leq \beta$ if $\alpha <_L \beta$ or $\alpha \subseteq \beta$.*
(iii) *$\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.*
(iv) *If $h \in [T]$, we say $\alpha <_L h$ ($h <_L \alpha$, $\alpha < h$, $h < \alpha$) if there exists $\beta \subset h$ such that $\alpha <_L \beta$ ($\beta <_L \alpha$, $\alpha < \beta$, $\beta < \alpha$, respectively).*

11.2. **Elements of M and $\widehat{\omega}$ on the tree.** We think of each element of M and each natural number as being painted on a ball. At each node α we place a pocket, called S_α, which can hold no more than a principal collection of M-balls, and a pocket called $\hat{S}_\alpha$ which can hold finitely-many $\widehat{\omega}$-balls. During the construction we pour balls into the tree, always starting from the top, S_λ ($\hat{S}_\lambda$). The balls will move on the tree, sometimes being retrieved to a higher pocket but in general moving downward. The $\widehat{\omega}$-ball marked $\hat{x}$ may move no lower than the level with nodes of length $\hat{x}$, and there may be other restrictions on the movement of $\hat{x}$. On the M side there are similar limitations on x, described for both M and $\widehat{\omega}$ in Steps 1 and 2 of the construction in §14. For $\alpha \subset f$, however, the collection of x ($\hat{x}$) which are not at or below α will be principal (finite).

The function $\alpha(x, s)$ ($\hat{\alpha}(\hat{x}, s)$) will designate the location of ball x ($\hat{x}$) at the end of stage s. We will guarantee in the construction that $\alpha(x) = \lim_s \alpha(x, s)$ ($\hat{\alpha}(\hat{x}) = \lim_s \hat{\alpha}(\hat{x}, s)$) exists. For each stage s we define

$$S_{\alpha,s} = \{x : \alpha(x, s) = \alpha\},$$
$$R_{\alpha,s} = \{x : \alpha(x, s) \supseteq \alpha\},$$
$$Y_{\alpha,s} = \bigcup\{R_{\alpha,t} : t \leq s\},$$

and likewise the hatted versions. The pocket S_α is called an *α-section*, and R_α an *α-region*. The region R_α consists of all elements in pockets at or below node α. We will prove that an element x can enter R_α at most once; however, it might not remain, so $R_{\alpha,\infty}$ (defined below) will be a d.c.e. set. Therefore we define the c.e. set $Y_\alpha = \bigcup_s Y_{\alpha,s}$ of all elements which are in R_α at any point during the construction. Another set we will find useful is

$$Y_{<\alpha} = \bigcup\{Y_\delta : \delta <_L \alpha\},$$

the collection of all elements which ever enter the pockets of nodes to the left of α.

Let $S_{\alpha,\infty} = \{x : \alpha(x) = \alpha\}$, and $R_{\alpha,\infty} = \{x : \alpha(x) \supseteq \alpha\}$. We will ensure that if $\alpha \subset f$, $R_{\alpha,\infty} =^\diamond Y_\alpha =^\diamond M$ ($\hat{R}_{\alpha,\infty} =^* \hat{Y}_\alpha =^* \omega$), by guaranteeing that $R_{\alpha,\infty}$ is empty if $f <_L \alpha$ and finite if $\alpha <_L f$, and that every $S_{\alpha,\infty}$ is principal or finite.

We will also guarantee that balls move into R_α from R_{α^-}, so that $Y_\alpha \setminus Y_{\alpha^-} = \emptyset$ (recall that $A \setminus B$ is $A - B$ together with the elements of $A \cap B$ which are enumerated into A before entering B). During the construction, the true path will be approximated by a computable sequence of finite strings $\{f_s\}_{s\in\omega}$, such that $f = \liminf_s f_s$. This approximation to the true path will restrict the movement of elements on the tree.

Definition 11.2. *If $f_s <_L \alpha$ at some stage s such that $x \lhd m_s$ ($\hat{x} \leq s$), the element x ($\hat{x}$) is* α-ineligible *at all stages $t \geq s$.*

If x is α-ineligible at stages $t \geq s$, we will require $x \notin S_{\alpha,t}$ ($\hat{x} \notin \hat{S}_{\alpha,t}$) for all $t \geq s$. The true path is defined in such a way that if $\alpha \subset f$, the number of times we see $f_s <_L \alpha$ is finite, so only a finite number of elements become α-ineligible.

11.3. **States and the duties of α.** Any given node α will either be building a U, $\widehat{U}$ pair, building a $\widehat{V}$, V pair, or thinking about non-well-resided α-states (Definition 11.3, below). Accordingly, we must spread out the U and V indices. Which nodes do what will depend on their length, so we assign to each node α indices e_α, $\hat{e}_\alpha$ which depend on $|\alpha|$. If α is building U_α, for instance, it will attempt to ensure $U_\alpha =^\diamond U_{e_\alpha}$. We begin by defining $e_\lambda = \hat{e}_\lambda = -1$, and continue inductively according to $|\alpha|$ as follows:

n	Activity at α for $\lvert\alpha\rvert = n \pmod 4$
0	Build U_α and $\widehat{U}_\alpha$ (goal: $\alpha \subset f \Rightarrow U_\alpha =^\diamond U_{e_\alpha}$) V_α, $\widehat{V}_\alpha$ undefined $e_\alpha = e_{\alpha^-} + 1$; $\hat{e}_\alpha = \hat{e}_{\alpha^-}$
1	Build $\widehat{V}_\alpha$ and V_α (goal: $\alpha \subset f \Rightarrow \widehat{V}_\alpha =^\diamond \widehat{V}_{e_\alpha}$) U_α, $\widehat{U}_\alpha$ undefined $e_\alpha = e_{\alpha^-}$; $\hat{e}_\alpha = \hat{e}_{\alpha^-} + 1$
2	Consider new α-states ν believed to be non-well-resided on Y_α (see §12.3) U_α, $\widehat{U}_\alpha$, V_α, $\widehat{V}_\alpha$ undefined $e_\alpha = e_{\alpha^-}$; $\hat{e}_\alpha = \hat{e}_{\alpha^-}$
3	Consider new α-states $\hat{\nu}$ believed to be non-well-resided on $\hat{Y}_\alpha$ (see §12.3) U_α, $\widehat{U}_\alpha$, V_α, $\widehat{V}_\alpha$ undefined $e_\alpha = e_{\alpha^-}$; $\hat{e}_\alpha = \hat{e}_{\alpha^-}$

Since it only makes sense to think about whether x is in U_α, say, when $|\alpha| = 0 \pmod 4$ (that is, when $e_\alpha = e_{\alpha^-} + 1$), we adjust our concept of e-state to α-state.

Definition 11.3. (i) *The* α-state *of x at stage s is*

$$\nu(\alpha, x, s) = \langle \alpha, \sigma(\alpha, x, s), \tau(\alpha, x, s)\rangle, \text{ where}$$

$$\sigma(\alpha, x, s) = \{e_\beta : \beta \subseteq \alpha \ \&\ e_\beta > e_{\beta^-} \ \&\ x \in U_{\beta,s}\}, \text{ and}$$

$$\tau(\alpha, x, s) = \{\hat{e}_\beta : \beta \subseteq \alpha \ \&\ \hat{e}_\beta > \hat{e}_{\beta^-} \ \&\ x \in \widehat{V}_{\beta,s}\}.$$

(ii) *The* final α-state *of x is*

$$\nu(\alpha, x) = \langle \alpha, \sigma(\alpha, x), \tau(\alpha, x)\rangle,$$

where $\sigma(\alpha, x) = \lim_s \sigma(\alpha, x, s)$ *and* $\tau(\alpha, x) = \lim_s \tau(\alpha, x, s)$.

(iii) *The only λ-state is* $\nu_{-1} = \langle \lambda, \emptyset, \emptyset\rangle$.

The α-state of $\hat{x}$ has the dual definition to the above.

For ease of discussion, we define some orderings and operations on states.

Definition 11.4. *Given α-states $\nu_0 = \langle \alpha, \sigma_0, \tau_0\rangle$ and $\nu_1 = \langle \alpha, \sigma_1, \tau_1\rangle$, we define the following inequalities, with the strict version of each defined as expected.*

(i) $\nu_0 \leq_B \nu_1$ *if* $\sigma_0 = \sigma_1$ *and* $\tau_0 \subseteq \tau_1$ *(BLUE claims more $\widehat{V}$ ideals).*
(ii) $\nu_0 \leq_R \nu_1$ *if* $\sigma_0 \subseteq \sigma_1$ *and* $\tau_0 = \tau_1$ *(RED claims more U ideals).*
(iii) $\hat{\nu}_0 \leq_B \hat{\nu}_1$ *if* $\hat{\sigma}_0 \subseteq \hat{\sigma}_1$ *and* $\hat{\tau}_0 = \hat{\tau}_1$ *(BLUE claims more $\widehat{U}$ sets).*
(iv) $\hat{\nu}_0 \leq_R \hat{\nu}_1$ *if* $\hat{\sigma}_0 = \hat{\sigma}_1$ *and* $\hat{\tau}_0 \subseteq \hat{\tau}_1$ *(RED claims more V sets).*

Note that considering $\hat{\nu}_0$ and $\hat{\nu}_1$ to be ν_0 and ν_1 read with respect to $\widehat{U}$ and V rather than U and $\widehat{V}$, we get the following correspondence:

$$[\nu_0 \leq_R \nu_1 \Leftrightarrow \hat{\nu}_0 \leq_B \hat{\nu}_1] \ \&\ [\nu_0 \leq_B \nu_1 \Leftrightarrow \hat{\nu}_0 \leq_R \hat{\nu}_1] \tag{11.1}$$

Definition 11.5. *Given $\alpha \in T$, $\beta \subseteq \alpha$, and an α-state $\nu_0 = \langle \alpha, \sigma_0, \tau_0\rangle$, a set of α-states $\mathcal{C}_\alpha$, or a finite set of α-states $\{\nu(\alpha, \sigma_i, \tau_i) : i \in I\}$:*

(i) $\nu_0 \restriction \beta = \langle \beta, \sigma_1, \tau_1\rangle$, *where* $\sigma_1 = \sigma_0 \cap \{0, \ldots, e_\beta\}$ *and* $\tau_1 = \tau_0 \cap \{0, \ldots, \hat{e}_\beta\}$.
(ii) $\mathcal{C}_\alpha \restriction \beta = \{\nu \restriction \beta : \nu \in \mathcal{C}_\alpha\}$.
(iii) $\nu_1 \preceq \nu_0$ *("ν_0 extends ν_1") if $\exists \beta$ such that* $\nu_0 \restriction \beta = \nu_1$.
(iv) $\bigcup\{\nu(\alpha, \sigma_i, \tau_i) : i \in I\} = \langle \alpha, \sigma, \tau\rangle$ *where* $\sigma = \bigcup\{\sigma_i : i \in I\}$ *and* $\tau = \bigcup\{\tau_i : i \in I\}$.

12. Keeping Track of the Residedness of States

12.1. **Well-visited states.** For each $\alpha \in T$ we define a number of sets of α-states. The set $\mathcal{F}_\alpha$ is the collection of α-states ν which are well-visited by elements x while they are in R_α. Adding the restriction that x must have the state ν when it first appears in R_α (which is to say, when it first appears in S_α) gives the set $\mathcal{E}_\alpha \subseteq \mathcal{F}_\alpha$. Each of these sets also has a dual. The explicit definitions are

$$\mathcal{E}_\alpha = \{\nu : (\exists^{np} x)(\exists s)[x \in S_{\alpha,s} - \bigcup\{S_{\alpha,t} : t < s\} \ \&\ \nu(\alpha, x, s) = \nu]\}$$

$$\mathcal{F}_\alpha = \{\nu : (\exists^{np} x)(\exists s)[x \in R_{\alpha,s} \ \&\ \nu(\alpha, x, s) = \nu]\}$$

where the duals are obtained by hatting appropriately and replacing $(\exists^{np} x)$ with $(\exists^\infty \hat{x})$.

To meet the automorphism requirement (10.1), we must have

$$\widehat{\mathcal{F}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{F}_\alpha\} \tag{12.1}$$

for $\alpha \subset f$. To achieve (12.1), each node α will also have an associated set $\mathcal{M}_\alpha$, the set of α-states α believes to be well-visited. At every node α we require $\mathcal{M}_\alpha \restriction \alpha^- = \mathcal{M}_{\alpha^-}$. For $\alpha \subset f$, we will prove that $\mathcal{M}_\alpha \subseteq \mathcal{E}_\alpha$ and $\mathcal{F}_\alpha \subseteq \mathcal{M}_\alpha$ to get $\mathcal{M}_\alpha = \mathcal{F}_\alpha = \mathcal{E}_\alpha$. Depending on the length of α, $\mathcal{F}_\alpha \subseteq \mathcal{M}_\alpha$ will either be proved directly or by proving the following three conditions.

$$\mathcal{E}_\alpha \subseteq \mathcal{M}_\alpha \tag{12.2}$$

(12.3)
(a.e. x)[if $x \in Y_{\alpha,s}, \nu_0 = \nu(\alpha, x, s) \in \mathcal{M}_\alpha,$ and BLUE causes enumeration of x so that $\nu(\alpha, x, s+1) = \nu_1,$ then $\nu_1 \in \mathcal{M}_\alpha$]

(12.4)
(a.e. x)[if $x \in Y_{\alpha,s}, \nu_0 = \nu(\alpha, x, s) \in \mathcal{M}_\alpha,$ and RED causes enumeration of x so that $\nu(\alpha, x, s+1) = \nu_1,$ then $\nu_1 \in \mathcal{M}_\alpha$]

Condition (12.2) will be met by exerting tight control of the entry of elements into S_α. Condition (12.3) will be met by ensuring $\mathcal{M}_\alpha$ is sufficiently closed with respect to BLUE's possible enumerations; that is, by making sure $\alpha \subset f$ is $\mathcal{M}$*-consistent*.

Definition 12.1. *A node α is $\mathcal{M}$-*inconsistent *if $e_\alpha > e_{\alpha^-}$ and there exist α-states $\nu_0 <_B \nu_1$ such that $\nu_0 \in \mathcal{M}_\alpha$, $\nu_1 \restriction \alpha^- \in \mathcal{M}_{\alpha^-}$, but $\nu_1 \notin \mathcal{M}_\alpha$. Otherwise α is $\mathcal{M}$-*consistent*.*

The dual notion is

Definition 12.2. *A node α is* $\widehat{\mathcal{M}}$-inconsistent *if $\hat{e}_\alpha > \hat{e}_{\alpha^-}$ and there exist α-states $\hat{\nu}_0 <_B \hat{\nu}_1$ such that $\hat{\nu}_0 \in \widehat{\mathcal{M}}_\alpha$, $\hat{\nu}_1 \restriction \alpha^- \in \widehat{\mathcal{M}}_{\alpha^-}$, but $\hat{\nu}_1 \notin \widehat{\mathcal{M}}_\alpha$. Otherwise α is* $\widehat{\mathcal{M}}$-consistent.

Condition (12.4) will be met via the dual case. By (12.2), $\mathcal{M}_\alpha$ contains many of the well-visited states: every one which is witnessed sufficiently by elements as they enter R_α. Together (12.3) and (12.4) guarantee that all of the states which are witnessed to be well-visited by elements which are already in R_α are also in $\mathcal{M}_\alpha$, giving $\mathcal{F}_\alpha \subseteq \mathcal{M}_\alpha$.

The dual $\widehat{\mathcal{M}}_\alpha$ is defined as

$$\widehat{\mathcal{M}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{M}_\alpha\}. \tag{12.5}$$

In the verification we will prove that $\widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}_\alpha = \widehat{\mathcal{E}}_\alpha$ as well, so that (12.1) is satisfied and the well-visited α-states coincide on the M and $\widehat{\omega}$ sides.

12.2. **Avoiding circularity.** Although the intention for $\mathcal{M}_\alpha$ is that it be equal to $\mathcal{F}_\alpha$, we must be able to determine from the node α^- which extension to take. Since $\mathcal{F}_\alpha$ is dependent on the particular α chosen, we now define a set which depends only on α^-. For $\beta = \alpha^-$, the new set $\mathcal{F}^+_\beta$ will be such that for $\alpha \subset f$, $\mathcal{M}_\alpha = \mathcal{F}^+_\beta = \mathcal{F}_\alpha$.

Fix $\alpha \in T$ such that $e_\alpha > e_\beta$ for $\beta = \alpha^-$. Define the c.e. set $Z_{e_\alpha} = \bigcup_s Z_{e_\alpha,s}$ where

$$Z_{e_\alpha,s+1} = \{x : x \in U_{e_\alpha,s+1} \;\&\; x \in Y_{\alpha^-,s}\}.$$

Define a new α-state $\nu^+(\alpha, x, s)$ exactly as for $\nu(\alpha, x, s)$ (Definition 11.3) but with $Z_{e_\alpha,s}$ in place of $U_{\alpha,s}$. Note that we are only changing (possibly) the last place of $\nu(\alpha, x, s)$. Define $\mathcal{F}^+_\beta$ and k^+_β as follows.

$$\mathcal{F}^+_\beta = \{\nu : (\exists^{np} x)(\exists s)[x \in Y_{\beta,s} \;\&\; \nu^+(\alpha, x, s) = \nu]\}. \tag{12.6}$$

$$\begin{aligned} k^+_\beta = \min\{y : (\forall x \rhd m_y)(\forall s)[[x \notin P_{<y} \;\&\; \\ x \in Y_{\beta,s} \;\&\; \nu^+(\alpha, x, s) = \nu_1] \longrightarrow \nu_1 \in \mathcal{F}^+_\beta]\}. \end{aligned} \tag{12.7}$$

The value k^+_β is the bound on the set of elements which have non-well-visited states (since there are only a finite number of α-states, only a principal collection of elements can have non-well-visited states). The object is to keep elements in $P_{<k^+_\beta}$ out of Y_α. We also define $\widehat{\mathcal{F}}^+_\beta = \{\hat{\nu} : \nu \in \mathcal{F}^+_\beta\}$. If $\alpha \in T$, $\beta = \alpha^-$ are such that $\hat{e}_\alpha > \hat{e}_\beta$, we define $\widehat{\mathcal{F}}^+_\beta$ and $\hat{k}^+_\beta$ using the duals to (12.6) and (12.7).

Along with $\mathcal{M}_\alpha$, every $\alpha \in T$ will have a k_α such that if $\alpha \subset f$, $k_\alpha = k^+_\beta$. If $e_\alpha = e_\beta$ and $\hat{e}_\alpha = \hat{e}_\beta$, we define $\mathcal{F}^+_\beta = \mathcal{F}_\beta$, $k^+_\beta = k_\beta$, and likewise for the duals. We allow x to enter Y_α only if $x \notin P_{<k_\alpha}$ (to

enter $\hat{Y}_\alpha$, $\hat{x}$ must be greater than $\hat{k}_\alpha$). Therefore if there is an element allowed into Y_α which has a state α considers non-well-visited, we have a witness that k_α is wrong.

Definition 12.3. *If* $(\exists x)(\exists s)[x \in Y_{\alpha,s}\ \&\ \nu(\alpha, x, s) \notin \mathcal{M}_\alpha]$*, then* α *is* provably incorrect *at all stages* $t \geq s$*.*

Nodes α which are provably incorrect are kept off the true path.

12.3. **Non-well-resided states.** As with well-visited states, we define several sets of states related to non-well-residedness for each node α. The set of non-well-resided α-states is

$$\mathcal{N}_\alpha = \{\nu_1 : \neg(\exists^{np} x)[x \in Y_\alpha\ \&\ \nu(\alpha, x) = \nu_1]\}.$$

Likewise we define $\widehat{\mathcal{N}}_\alpha$. As with the well-visited states in requirement (12.1), we must show for all $\alpha \subset f$ that

$$\widehat{\mathcal{N}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{N}_\alpha\}. \tag{12.8}$$

While $\mathcal{F}_\alpha$ and $\mathcal{E}_\alpha$ are Π^0_2, and so can be guessed at (almost) directly in the construction, $\mathcal{N}_\alpha$ is Σ^0_3 and so requires approximation. The Π^0_2 approximation will be the disjoint union of two sets $\mathcal{R}_\alpha$ and $\mathcal{B}_\alpha$, which correspond to states α believes are non-well-resided and emptied by RED or BLUE respectively.

We define $\mathcal{R}_\alpha$, $\mathcal{B}_\alpha$, and their duals inductively. Fix $\alpha \in T$ and assume $\mathcal{R}_\gamma$, $\mathcal{B}_\gamma$, $\widehat{\mathcal{R}}_\gamma$, and $\widehat{\mathcal{B}}_\gamma$ have been defined for all $\gamma \subset \alpha$. We define all four sets as disjoint unions, e.g.,

$$\mathcal{R}_\alpha = \mathcal{R}^\alpha_\alpha \sqcup \mathcal{R}^{<\alpha}_\alpha.$$

Define

$$\mathcal{R}^{<\alpha}_\alpha = \{\nu : \nu \in \mathcal{M}_\alpha\ \&\ \nu \restriction \alpha^- \in \mathcal{R}_{\alpha^-}\}.$$

The set $\mathcal{B}^{<\alpha}_\alpha$ is defined as above but with $\mathcal{B}_{\alpha^-}$ in place of $\mathcal{R}_{\alpha^-}$, and $\widehat{\mathcal{B}}^{<\alpha}_\alpha$ and $\widehat{\mathcal{R}}^{<\alpha}_\alpha$ are defined likewise, with appropriate hatting. If $|\alpha| \not\equiv 2 \pmod 4$, we set

$$\mathcal{R}^\alpha_\alpha = \widehat{\mathcal{B}}^\alpha_\alpha = \emptyset;$$

they might be nonempty otherwise. Note that when $|\alpha| \equiv 2 \pmod 4$, $\mathcal{R}^{<\alpha}_\alpha$ depends only on nodes up to α^- because at such an α, $e_\alpha = e_{\alpha^-}$ and $\hat{e}_\alpha = \hat{e}_{\alpha^-}$, so $\mathcal{M}_\alpha = \mathcal{M}_{\alpha^-}$.

If $|\alpha| \equiv 2 \pmod 4$, we define the Π^0_2 predicate

$$F(\alpha^-, \nu) \equiv (\forall x)[x \in Y_{\alpha^-} \longrightarrow (\nu(\alpha, x) \neq \nu\ \vee\ x \in P_{\leq|\alpha^-|})].$$

$F(\alpha^-, \nu)$ says that any element with state ν at the end of the construction is in the ideal generated by the first $|\alpha^-|$ elements of M enumerated. That is, α^- witnesses that ν corresponds to a principal ideal and is thus non-well-resided. Note also that as with $\mathcal{F}_\beta^+$, $F(\alpha^-, \nu)$ avoids circularity, since α-state depends only on $|\alpha|$. Having defined $F(\alpha^-, \nu)$, we let $\mathcal{R}_\alpha^\alpha$ be nonempty, allowing $\alpha \subset f$ only if

$$\mathcal{R}_\alpha^\alpha = \{\nu : \nu \in \mathcal{M}_\alpha - (\mathcal{R}_\alpha^{<\alpha} \cup \mathcal{B}_\alpha^{<\alpha}) \;\&\; F(\alpha^-, \nu)\}.$$

Also for $|\alpha| \equiv 2 \pmod 4$, we define

$$\widehat{\mathcal{B}}_\alpha^\alpha = \{\hat{\nu} : \nu \in \mathcal{R}_\alpha^\alpha\}.$$

If $|\alpha| \not\equiv 3 \pmod 4$, we set

$$\widehat{\mathcal{R}}_\alpha^\alpha = \mathcal{B}_\alpha^\alpha = \emptyset.$$

If $|\alpha| \equiv 3 \pmod 4$, we allow $\widehat{\mathcal{R}}_\alpha^\alpha \neq \emptyset$, defining the predicate $\hat{F}(\alpha^-, \hat{\nu})$ as follows.

$$\hat{F}(\alpha^-, \hat{\nu}) \equiv (\forall \hat{x})[[\hat{x} > |\alpha^-| \;\&\; \hat{x} \in \hat{Y}_{\alpha^-}] \;\rightarrow\; \hat{\nu}(\alpha, \hat{x}) \neq \hat{\nu}]$$

Again, the requirement is that for $\alpha \subset f$,

$$\widehat{\mathcal{R}}_\alpha^\alpha = \{\hat{\nu} : \hat{\nu} \in \widehat{\mathcal{M}}_\alpha - (\widehat{\mathcal{R}}_\alpha^{<\alpha} \cup \widehat{\mathcal{B}}_\alpha^{<\alpha}) \;\&\; \hat{F}(\alpha^-, \hat{\nu})\}$$

and we define

$$\mathcal{B}_\alpha^\alpha = \{\nu : \hat{\nu} \in \widehat{\mathcal{R}}_\alpha^\alpha\}.$$

It will be BLUE's responsibility to change the state of elements x such that $\nu(\alpha, x, s) \in \mathcal{B}_\alpha$, for $x \in R_\alpha$, which takes care of half of the approximation. For $\mathcal{R}_\alpha$, we know that if $\alpha \subset f$, $\mathcal{R}_\alpha$ will in fact be non-well-resided, so

$$(\forall \nu \in \mathcal{R}_\alpha)(\text{a.e. } x \in Y_\alpha)(\forall s)[\nu(\alpha, x, s) = \nu \longrightarrow (\exists t > s)[\nu(\alpha, x, t) \neq \nu]]. \tag{12.9}$$

Therefore BLUE can wait for RED to move elements out of states in $\mathcal{R}_\alpha$. This leads to the definition of another kind of consistency. Since $\alpha \subset f$ means that all states in $\mathcal{R}_\alpha$ must be emptied by RED, for every state in $\mathcal{R}_\alpha$ there must be a state reachable in RED moves which is not non-well-resided. Furthermore, since there are only a finite number of α-states, at least one such state must also be well-visited. This is another closure property of $\mathcal{M}_\alpha$, as was $\mathcal{M}$-consistency.

Definition 12.4. *A node $\alpha \in T$ is* $\mathcal{R}$-consistent *if*

$$(\forall \nu_0 \in \mathcal{R}_\alpha)(\exists \nu_1 \in \mathcal{M}_\alpha)[\nu_0 <_R \nu_1]$$

and $\mathcal{R}$-inconsistent *otherwise.*

The dual notion is

Definition 12.5. *A node* $\alpha \in T$ *is* $\widehat{\mathcal{R}}$-consistent *if*

$$(\forall \hat{\nu}_0 \in \widehat{\mathcal{R}}_\alpha)(\exists \hat{\nu}_1 \in \widehat{\mathcal{M}_\alpha})[\hat{\nu}_0 <_R \hat{\nu}_1]$$

and $\widehat{\mathcal{R}}$-inconsistent *otherwise.*

As with $\mathcal{M}$-consistency, we will require $\alpha \subset f$ to be $\mathcal{R}$-consistent. Therefore for $\alpha \subset f$, using (11.1) and the definition of $\widehat{\mathcal{B}}_\alpha$ we know

$$(\forall \hat{\nu}_0 \in \widehat{\mathcal{B}}_\alpha)(\exists \hat{\nu}_1 \in \widehat{\mathcal{M}_\alpha})[\hat{\nu}_0 <_B \hat{\nu}_1]. \tag{12.10}$$

Note that Equation (12.10) and Definition 12.4, via repeated application, guarantee after some number of iterations of moves by BLUE or RED we can get out of $\widehat{\mathcal{B}}_\alpha$ or $\mathcal{R}_\alpha$, respectively. Thus, for every state α believes to be emptied by BLUE, there must be a state which α believes to be well-visited and not emptied by BLUE which is reachable by the BLUE moves. That motivates the following definition.

Definition 12.6. *A function* $\hat{h}_\alpha : \widehat{\mathcal{B}}_\alpha \to (\widehat{\mathcal{M}_\alpha} - \widehat{\mathcal{B}}_\alpha)$ *is a* target function *if*

$$(\forall \hat{\nu} \in \widehat{\mathcal{B}}_\alpha)[\hat{\nu} <_B \hat{h}_\alpha(\hat{\nu})].$$

Dually, $h_\alpha : \mathcal{B}_\alpha \to (\mathcal{M}_\alpha - \mathcal{B}_\alpha)$ *is a target function if*

$$(\forall \nu \in \mathcal{B}_\alpha)[\nu <_B h_\alpha(\nu)].$$

The notes preceding the definition assert the existence of such an h_α for $\alpha \subset f$. We will require that for almost every $x \in \mathcal{B}_\alpha$, BLUE must move x to the target state $h_\alpha(\nu(\alpha, x, s))$.

Since $\mathcal{R}_\alpha$ and $\mathcal{B}_\alpha$ are approximations, we must make sure that using them we empty exactly the states in $\mathcal{N}_\alpha$. By the use of $F(\alpha^-, x)$ in the definition of $\mathcal{R}_\alpha$, we know $\mathcal{R}_\alpha \cup \mathcal{B}_\alpha \subseteq \mathcal{N}_\alpha$. In order to guarantee we empty all states in $\mathcal{N}_\alpha$, it is sufficient to make sure that if $\alpha \subset f$ and $\nu_0 \in \mathcal{N}_\alpha$, there is some $\gamma \supseteq \alpha$ such that $\gamma \subset f$ and for all $\nu_1 \in \mathcal{M}_\gamma$ which extend ν_0, $\nu_1 \in \mathcal{R}_\gamma \cup \mathcal{B}_\gamma$. Removing references to $\mathcal{N}_\alpha$, the statement we must prove is

$$\begin{aligned} &(\forall \alpha \subset f)(\forall \nu_0 \in \mathcal{M}_\alpha)(\neg \exists^{np} x)[x \in Y_\alpha \ \&\ \nu(\alpha, x) = \nu_0] \Longrightarrow \\ &\quad (\exists \gamma)[\alpha \subseteq \gamma \subset f \ \&\ \{\nu_1 \in \mathcal{M}_\gamma : \nu_1 \restriction \alpha = \nu_0\} \subseteq \mathcal{R}_\gamma \cup \mathcal{B}_\gamma] \end{aligned} \tag{12.11}$$

along with its dual. To check this, fix some $\alpha \subset f$ and $\nu_0 \in \mathcal{M}_\alpha$ such that the hypothesis of (12.11) holds. Since $\alpha \subset f$ we know $Y_\alpha =^\diamond M$, so we can find some i such that for all $x \in M$, $x \notin P_{\leq i} \Rightarrow \nu(\alpha, x) \neq \nu_0$. Choose $\gamma \subset f$ such that $\alpha \subseteq \gamma$, $|\gamma| > i$, and $|\gamma| \equiv 2 \pmod 4$. Consider any $\nu_1 \in \mathcal{M}_\gamma$ such that $\nu_1 \restriction \alpha = \nu_0$. If ν_1 is not in $\mathcal{R}_\gamma^{<\gamma} \cup \mathcal{B}_\gamma^{<\gamma}$, then $F(\gamma^-, \nu_1)$ holds, so by definition of $\mathcal{R}_\gamma^\gamma$ for $\gamma \subset f$, $\nu_1 \in \mathcal{R}_\gamma^\gamma$. The dual statement is proved likewise.

Finally, we note that since $\mathcal{B}_\alpha$ and $\widehat{\mathcal{B}}_\alpha$ are defined as duals to $\mathcal{R}_\alpha$ and $\widehat{\mathcal{R}}_\alpha$, again using (11.1), to show all of the states in these four sets are emptied it suffices to prove

$$(\forall \nu_0 \in \mathcal{B}_\alpha)[\{x : \nu(\alpha, x) = \nu_0\} =^\diamond \emptyset] \tag{12.12}$$

and its dual.

13. The Definition of the Tree and the True Path

First we collect our notions of consistency, allowing a node on the tree to have successors only if it satisfies all such notions.

Definition 13.1. *A node $\alpha \in T$ is* consistent *if it is $\mathcal{M}$-, $\widehat{\mathcal{M}}$-, $\mathcal{R}$-, and $\widehat{\mathcal{R}}$-consistent.*

In the following definition, the intended meanings of $\mathcal{M}_\alpha$, $\mathcal{R}_\alpha$, $\mathcal{B}_\alpha$, and k_α have already been explained. The number $c_\alpha \in \omega$ is an additional empty symbol that will guess a Σ^0_3 predicate; its function is explained below, in Definition 13.5 and the remarks that follow it.

Definition 13.2 (T, the construction tree)**.** *Put $\lambda \in T$, and let $\mathcal{M}_\lambda$, $\mathcal{R}_\lambda$, and $\mathcal{B}_\lambda$ all be empty. Define $k_\lambda = e_\lambda = \hat{e}_\lambda = -1$. If $\beta \in T$, put $\alpha = \beta^\frown\langle \mathcal{M}_\alpha, \mathcal{R}_\alpha, \mathcal{B}_\alpha, k_\alpha, c_\alpha\rangle$ in T provided it meets the following conditions:*

(i) *β is consistent.*
(ii) *$\mathcal{M}_\alpha$ is a set of α-states; $\mathcal{R}_\alpha, \mathcal{B}_\alpha \subseteq \mathcal{M}_\alpha$; $\mathcal{R}_\alpha \cap \mathcal{B}_\alpha = \emptyset$.*
(iii) *$\mathcal{M}_\alpha \upharpoonright \beta = \mathcal{M}_\beta$.*
(iv) *$(e_\alpha = e_{\alpha^-}$ & $\hat{e}_\alpha = \hat{e}_{\alpha^-}) \Rightarrow \mathcal{M}_\alpha = \mathcal{M}_\beta$.*
(v) *$\mathcal{R}^{<\alpha}_\alpha \subseteq \mathcal{R}_\alpha$; $\mathcal{B}^{<\alpha}_\alpha \subseteq \mathcal{B}_\alpha$.*
(vi) *$\mathcal{R}^\alpha_\alpha \neq \emptyset \Rightarrow |\alpha| \equiv 2 \ (mod\ 4)$; $\mathcal{B}^\alpha_\alpha \neq \emptyset \Rightarrow |\alpha| \equiv 3 \ (mod\ 4)$.*

In addition, each $\alpha \in T$ has associated dual sets $\widehat{\mathcal{M}}_\alpha$, $\widehat{\mathcal{R}}_\alpha$, and $\widehat{\mathcal{B}}_\alpha$, determined from $\mathcal{M}_\alpha$, $\mathcal{R}_\alpha$, and $\mathcal{B}_\alpha$, respectively, as well as integers e_α and $\hat{e}_\alpha$ depending only on $|\alpha|$. Recall that we are associating $\langle \mathcal{M}_\alpha, \mathcal{R}_\alpha, \mathcal{B}_\alpha, k_\alpha, c_\alpha\rangle$ with an integer under some effective coding so that we may regard T as a subset of $\omega^{<\omega}$.

Definition 13.3. *The* true path *$f \in [T]$ is defined by induction on n. If $\beta = f \upharpoonright (n-1)$ has been defined and is consistent, then $f \upharpoonright n$ is the $<_L$-least length-n extension α of β such that the following hold:*

(i) *$n \equiv 0 \ (mod\ 4) \Longrightarrow \mathcal{M}_\alpha = \mathcal{F}^+_\beta$ and $k_\alpha = k^+_\beta$.*
(ii) *$n \equiv 1 \ (mod\ 4) \Longrightarrow \widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}^+_\beta$ and $k_\alpha = k^+_\beta$.*
(iii) *$n \equiv 2 \ (mod\ 4) \Longrightarrow \begin{array}{c}\mathcal{R}^\alpha_\alpha = \{\nu : \nu \in \mathcal{M}_\alpha - (\mathcal{R}^{<\alpha}_\alpha \cap \mathcal{B}^{<\alpha}_\alpha) \ \& \ F(\beta, \nu)\} \\ \text{and } \widehat{\mathcal{B}}^\alpha_\alpha = \{\hat{\nu} : \nu \in \mathcal{R}^\alpha_\alpha\}.\end{array}$*

(iv) $n \equiv 3\ (mod\ 4) \Longrightarrow \begin{array}{c} \widehat{\mathcal{R}}^\alpha_\alpha = \{\hat{\nu} : \hat{\nu} \in \widehat{\mathcal{M}}_\alpha - (\widehat{\mathcal{R}}^{<\alpha}_\alpha \cap \widehat{\mathcal{B}}^{<\alpha}_\alpha)\ \&\ \hat{F}(\beta, \nu)\} \\ \text{and } \mathcal{B}^\alpha_\alpha = \{\nu : \hat{\nu} \in \widehat{\mathcal{R}}^\alpha_\alpha\}. \end{array}$

(v) *Unless otherwise specified above, $\mathcal{M}_\alpha$, $\mathcal{R}_\alpha$, $\mathcal{B}_\alpha$, and k_α have the values $\mathcal{M}_\beta$, $\mathcal{R}_\beta$, $\mathcal{B}_\beta$, and k_β, respectively, as in Definition 13.2.*

(vi) *The set C_α, defined below in Definition 13.5, is infinite.*

For a consistent $\beta = f \upharpoonright n$, note that $\mathcal{F}^+_\beta$ is just a finite set of states and k^+_β is an integer, so we may find α satisfying Conditions (i)-(v) of the definition. In fact, it is clear that there are unique $\mathcal{M}_\alpha$ and k_α satisfying the conditions. To see the same for $\mathcal{R}_\alpha$, recall from §12.3 that $\mathcal{R}_\alpha = \mathcal{R}^\alpha_\alpha \sqcup \mathcal{R}^{<\alpha}_\alpha$, where $\mathcal{R}^{<\alpha}_\alpha$ depends only on β, and $\mathcal{R}^\alpha_\alpha$ is uniquely determined by Conditions (iii) and (v). Likewise, $\mathcal{B}_\alpha$ is uniquely determined by Conditions (iv) and (v). We will show that of the α meeting (i)-(v), there is a unique α meeting (vi) (see Definition 13.5 and the remarks that follow). Hence, as long as every node on f is consistent, which will be proved in Lemmas 15.10 and 15.12, f is infinite.

Condition (vi) of Definition 13.3 is included so we may approximate the true path during the isomorphism construction. We will now define C_α. Recalling the remarks in §10 and §12, we see that Conditions (i)-(v) of Definition 13.3 are uniformly Δ^0_3 in β, and thus also uniformly Σ^0_3 in β. The following lemma is a modification of Lemma 2.35 in Cholak [6], which is an easy modification of Theorem IV.3.4 in Soare [19]. Define $\mathcal{A}$ to be the following set:

$$\{(\alpha, \beta) : \alpha \text{ satisfies Conditions (i)-(v) of Definition 13.3 w.r.t. } \beta\}.$$

The set $\mathcal{A}$ is uniformly Δ^0_3 and hence Σ^0_3.

Lemma 13.4. *Let $\mathcal{A}$ be defined as above. Since $\mathcal{A}$ is Σ^0_3, then there is a computable function g such that*

$$x \in \mathcal{A} \iff (\exists! c)[|W_{g(x,c)}| = \infty]$$

and

$$x \notin \mathcal{A} \iff (\forall c)[|W_{g(x,c)}| < \infty].$$

Definition 13.5. *Let g be the function given by Lemma 13.4. For $x = (\alpha, \beta)$, where $\alpha = \beta^\frown\langle \mathcal{M}_\alpha, \mathcal{R}_\alpha, \mathcal{B}_\alpha, k_\alpha, c_\alpha\rangle$, the* chip set *$C_\alpha$ is the set $W_{g(x,c_\alpha)}$.*

We will use the chip sets in §14, Step 6A, to define the true path approximation, a computable sequence of finite strings $\{f_s\}_{s\in\omega}$ such that $f = \liminf_s f_s$. For any consistent β, there are unique $\mathcal{M}_\alpha$, $\mathcal{R}_\alpha$, $\mathcal{B}_\alpha$, and k_α satisfying Conditions (i)-(v) of Definition 13.3, and hence there is a unique α such that C_α is infinite. Therefore the chip sets form a computable sequence of c.e. sets, $\{C_\alpha\}_{\alpha\in T}$, such that $\alpha \subset f$ iff $\beta = \alpha^-$ is on the true path and $|C_\alpha| = \infty$.

We included c_α in the node α so we could attach a particular chip set to each node of the tree. Once c_α is included in the node, there are an infinite number of paths through the tree that satisfy Conditions (i)-(v) of Definition 13.3 at every level. Condition (vi) is then included to ensure the uniqueness of the true path f.

Given the sequence $\{C_\alpha\}_{\alpha\in T}$, fix a simultaneous computable enumeration $\{C_{\alpha,s}\}_{\alpha\in T, s\in\omega}$ for use in §14, Step 6A.

To ensure $\mathcal{M}_\alpha \subseteq \mathcal{E}_\alpha$, we define $\mathcal{L}$, a list of elements of the form $\langle \alpha, \nu_1\rangle$, such that $\nu_1 \in \mathcal{M}_\alpha$. Loosely speaking, we allow an element x into $S_{\alpha,s+1}$ only when there is an unused entry $\langle \alpha, \nu_1\rangle \in \mathcal{L}_s$ such that x may be enumerated in such a way as to give $\nu(\alpha, x, s+1) = \nu_1$. In such a case we *mark* the entry $\langle \alpha, \nu_1\rangle$. $\mathcal{L}_s$ is augmented with new elements beginning with α at any stage s such that it and $\widehat{\mathcal{L}}_s$ are both α-*marked*; that is, all entries of the form $\langle \alpha, \nu_1\rangle$ on $\mathcal{L}$ ($\langle \alpha, \hat{\nu}_1\rangle$ on $\widehat{\mathcal{L}}_s$) have been marked. The value $m(\alpha, s)$ is the number of times $\mathcal{L}$ and $\widehat{\mathcal{L}}$ have been α-marked by the end of stage s. It does not have a hatted version.

14. The Construction

Steps 1-5 below, their duals $\hat{1}$-$\hat{5}$, and a final Step 6, produce the isomorphism. The duals should be clear; in cases where there may be ambiguity, it is explicitly noted. The "purpose of" remarks after some steps may contain statements to be proved in §15. There is one remaining definition we need for the construction.

Definition 14.1. *To* initialize *a node α means to remove every $x \in S_{\alpha,s}$ ($\hat{x} \in \hat{S}_{\alpha,s}$), and put x into $S_{\beta,s}$ ($\hat{x}$ into $\hat{S}_{\beta,s}$) for $\beta = \alpha \cap f_{s+1}$.*

Stage s=0: For all $\alpha \in T$ define $U_{\alpha,0} = V_{\alpha,0} = \widehat{U}_{\alpha,0} = \widehat{V}_{\alpha,0} = \emptyset$ and $m(\alpha, 0) = 0$. Define $Y_{\lambda,0} = Y_{\lambda,0} = \emptyset$ and $f_0 = \lambda$.

Stage s+1: Find the least $n < 6$ such that Step n's hypotheses are satisfied for some $x \in Y_{\alpha,s}$ and perform Step n's action. If there is no such n, find the least $n < 6$ such that some Step $\hat{n}$ applies. If all of those fail to apply, apply Step 6. At the end of every step, close all ideals U_α, $\widehat{V}_\alpha$ with respect to $Y_{\lambda,s+1}$.

In the following steps, let $x \in Y_{\lambda,s}$ ($\hat{x} \in \hat{Y}_{\lambda,s}$) and $\alpha \in T$, $\alpha \neq \lambda$, be arbitrary, and let $\beta = \alpha^-$. Recall that $x \lhd y$ means in the fixed enumeration of M, x is enumerated before y. $P_{\unlhd x}$ is the ideal generated by all elements of M enumerated up to and including x, and $P_{\lhd x}$ is the ideal generated by all elements of M enumerated up to but not including x. Letting $M = \{m_0, \ldots, m_i, \ldots\}$, we have the shorthand $P_{<i} := P_{\lhd m_i}$ and $P_{\leq i} := P_{\unlhd m_i}$.

Step 1: Let $\langle \alpha, \nu_1 \rangle$ be the first unmarked entry of $\mathcal{L}$ ($\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$; note that by the definition of $\mathcal{L}$, $\nu_1 \in \mathcal{M}_\alpha$). Look for x meeting the following conditions.

Size

1: $x \notin P_{<k_\alpha}$ (in $\hat{1}$, $\hat{x} > \hat{k}_\alpha$), $x \rhd m_{|\alpha|}$
2: x is α-eligible
3: $x \rhd m_{m(\alpha,s)}$

Location

4: $x \in R_{\beta,s} - Y_{\alpha,s}$
5: $\neg(\alpha(x,s) <_L \alpha)$

State and Independence

6: $\nu(\beta, x, s) = \nu_1 \upharpoonright \beta$
7: $e_\alpha > e_\beta \Rightarrow \nu^+(\alpha, x, s) = \nu_1$
8: $x \notin \langle Y_{\alpha,s} \rangle$ (absent from $\hat{1}$)

Choose the least such x (with respect to $\lhd$) and perform the following actions.

9: mark the list entry $\langle \alpha, \nu_1 \rangle$.
10: put x into S_α
11: if $e_\alpha > e_\beta$ and $e_\alpha \in \sigma_1$, then put x into $U_{\alpha,s+1}$
12: if $\hat{e}_\alpha > \hat{e}_\beta$ and $\hat{e}_\alpha \in \tau_1$, then put x into $\widehat{V}_{\alpha,s+1}$

Purpose of Step 1: If $\alpha \subset f$ and $\nu_1 \in \mathcal{M}_\alpha$, then $\mathcal{L}$ will have an infinite number of entries of the form $\langle \alpha, \nu_1 \rangle$ put on it and later marked. Each time such an entry is marked, an element x, which is not in the principal ideal of $Y_{\alpha,s}$, is put into S_α for the first time and given state ν_1. Since that happens an infinite number of times, ν_1 is well-visited by independent elements when they first appear in S_α; i.e., $\nu_1 \in \mathcal{E}_\alpha$ and $\mathcal{M}_\alpha \subseteq \mathcal{E}_\alpha$.

Step 2: Look for x and α meeting the following conditions.

1: $x \in S_{\beta,s}$
2: $x \rhd m_{|\alpha|}$, $x \notin P_{<k_\alpha}$ (in $\hat{1}$, $\hat{x} > \hat{k}_\alpha$)
3: x is α-eligible
4: $x \lhd m_{m(\alpha,s)}$ (contrast with 1.3)
5: α is the leftmost $\gamma \in T$ such that when you put γ in for α and γ^- in for β, 2.1-2.4 are satisfied

Choose the least such pair $\langle \alpha, x \rangle$ (with respect to node length and $\lhd$) and move x from S_β to S_α.

Purpose of Step 2: If $\alpha \subset f$, this ensures $R_\alpha =^\diamond M$ ($\hat{R}_\alpha =^* \omega$). We control with condition 2.4 in order to slow the flow down the tree. This keeps us from pouring too many elements down a path which is not f; $m(\alpha, s) \to \infty$ iff $\alpha \subset f$, so this bounds how much may move down into nodes α which are not on f.

Step 3: Look for x and α meeting the following conditions.

1: $e_\alpha > e_\beta$
2: $x \in S_{\alpha,s}$
3: $\nu(\alpha, x, s) = \nu_0 \in \mathcal{M}_\alpha$
4: $(\exists \nu_1)[\nu_0 <_B \nu_1 \;\&\; \nu_1 \upharpoonright \beta \in \mathcal{M}_\beta \;\&\; \nu_1 \notin \mathcal{M}_\alpha]$

Choose the least such pair $\langle \alpha, x\rangle$ (with respect to node length and $\lhd$) and enumerate x into $\widehat{V}_{\delta,s+1}$ for all $\delta \subset \alpha$ such that $e_\delta \in \tau_1$.

Purpose of Step 3: If α is $\mathcal{M}$-inconsistent (which means exactly conditions 3.1, 3.3, 3.4), witnessed by $x \in S_\alpha$ (condition 3.2), then we give x the state ν_1 to make α provably incorrect (which means there is an element in the region, in particular, x, which has a state α considers non-well-visited). This knocks α off of f.

Step 4: Look for $x \in R_{\alpha,s}$ meeting the following conditions.

1: $e_\alpha > e_\beta$
2: $x \notin U_{\alpha,s}$
3: $x \in Z_{e_\alpha,s}$

Choose the least such pair $\langle \alpha, x\rangle$ (with respect to node length and $\lhd$) and enumerate x into $U_{\alpha,s+1}$.

Step 5: Look for x and α satisfying the conditions of one of the following two cases.

Case 1

1: $\nu(\alpha, x, s) = \nu_0 \in \mathcal{B}_\alpha$, say $\nu_0 = \langle \alpha, \sigma_0, \tau_0\rangle$
2: $x \in S_{\alpha,s}$
3: α is $\mathcal{M}$-consistent and $\mathcal{R}$-consistent

Case 2

1: $\nu(\alpha, x, s) = \nu_0 \in \mathcal{B}_\alpha$, say $\nu_0 = \langle \alpha, \sigma_0, \tau_0\rangle$
4: $x \in S_{\delta,s}$ where $\delta^- = \alpha$
5: δ is either $\mathcal{M}$-inconsistent or $\mathcal{R}$-inconsistent

In either case, choose the least such $\langle \alpha, x\rangle$ (with respect to node length and $\lhd$). We want to progress toward emptying $\mathcal{B}_\alpha$ by changing x's state. Let $\nu_1 = h_\alpha(\nu_0)$, which will be BLUE-greater than ν_0 (by definition of h_α), $\nu_1 = \langle \alpha, \sigma_1, \tau_1\rangle$. Enumerate x into $\widehat{V}_\gamma$ for all $\gamma \subseteq \alpha$ such that $\hat{e}_\gamma > \hat{e}_{\gamma^-}$ and $e_\gamma \in \tau_1 - \tau_0$ (that is, we considered a new $\widehat{V}$ set at γ and it was in the chosen extension to x's α-state). This makes $\nu(\alpha, x, s+1) = \nu_1$.

Step 6:

6A: Define δ_t by induction for $t \leq s+1$. Let $\delta_0 = \lambda$. Given δ_t, let $v \leq s$ be maximal such that $\delta_t \subseteq f_v$ if such a v exists, and let $v = 0$ otherwise (v is the most recent stage at which the true path appeared to go through δ_t). Choose the $<_L$-least $\alpha \in T$ such that $\alpha^- = \delta_t$ and $C_{\alpha,s} \neq C_{\alpha,v}$. If such an α exists, define $\delta_{t+1} = \alpha$. If not, define $\delta_{t+1} = \delta_t$. That is, look for the leftmost node which extends δ_t by one and which has increased its chip set since the last time you were at that node (because $\alpha \subset f \iff |C_\alpha| = \infty$) and go through there.

Define $f_{s+1} = \delta_{s+1}$.

6B: For every $\alpha \subseteq f_{s+1}$ such that both $\mathcal{L}_s$ and $\widehat{\mathcal{L}}_s$ are α-marked (every entry beginning with α is marked), do the following:

1: define $m(\alpha, s+1) = m(\alpha, s) + 1$

2: add to the bottom of $\mathcal{L}_s$ a new unmarked α-entry $\langle \alpha, \nu \rangle$ for every $\nu \in \mathcal{M}_\alpha$. Do likewise for $\widehat{\mathcal{L}}_s$.

After doing the above for all relevant α, let $\mathcal{L}_{s+1}$ be the augmented $\mathcal{L}_s$ and likewise for $\widehat{\mathcal{L}}_{s+1}$. If no such α exists, let the stage $s+1$ version of everything equal the stage s version.

6C: Empty R_α to the right of f_{s+1}: initialize all α to the right of f_{s+1}. That is, pull all the balls in α's pockets up to where α branches off from f_{s+1}.

6D: Add balls to the machine: choose the $\lhd$-least $x \notin Y_{\lambda,s}$ such that $x \lhd m_s$ ($<$-least $\hat{x} \notin \hat{Y}_{\lambda,s}$ such that $x < s$) and put x into S_λ ($\hat{x}$ into $\hat{S}_\lambda$). For each $x \in Y_{\lambda,s+1}$, let $\alpha(x, s+1)$ denote the unique γ such that $x \in S_{\gamma,s+1}$, and likewise for all $\hat{x}$.

15. The Isomorphism Theorem and Verification

Theorem 15.1 (Isomorphism Theorem)**.** *Suppose c.e. ideals $\{U_\alpha\}_{\alpha\in T}$ and $\{\widehat{V}_\alpha\}_{\alpha\in T}$ and c.e. sets $\{\widehat{U}_\alpha\}_{\alpha\in T}$ and $\{V_\alpha\}_{\alpha\in T}$ are enumerated by the construction in §14 using Steps 1-5, $\hat{1}$-$\hat{5}$, and 6. Then the correspondence $U_\alpha \leftrightarrow \widehat{U}_\alpha$, $\widehat{V}_\alpha \leftrightarrow V_\alpha$, $\alpha \subset f$, defines an isomorphism between $G^\diamondsuit$ and $\mathcal{E}^*$.*

The proof of the theorem is split into the following thirteen lemmas. Lemmas 15.3, 15.7, 15.8, 15.11, and 15.14 have no duals. The remaining lemmas have a dual case whose proof should be clear from the proof as written.

Lemma 15.2. *At stage $s+1$,*

(i) *if x enters R_α, $\alpha \neq \lambda$, it is via Step 1 or Step 2 applying to α and x;*
(ii) *if x moves from S_α to S_δ, it is via one of the following three steps:*
  (a) *Step 1 applies to δ and x ($\delta <_L \alpha$ or $\delta^- = \alpha$);*
  (b) *Step 2 applies to δ and x ($\delta^- = \alpha$);*
  (c) *Step 6C applies to α ($f_{s+1} <_L \alpha$);*
(iii) *if $x \in S_{\alpha,s}$ is enumerated in a RED set $U_{\alpha,s+1}$ such that x is not generated by the elements in $U_{\alpha,s}$, it is via Step 1 or Step 4 applying to α and x;*
(iv) *if $x \in S_{\alpha,s}$ is enumerated in a BLUE set $\widehat{V}_{\alpha,s+1}$ such that x is not generated by the elements in $\widehat{V}_{\alpha,s}$, it is via one of the following three steps:*
  (a) *Step 1 applies to x and α ($\hat{e}_\alpha > \hat{e}_\beta$);*
  (b) *Step 3 applies to x and some $\delta \supset \alpha$;*
  (c) *Step 5 applies to x and some $\delta \supseteq \alpha$ ($\hat{e}_\alpha > \hat{e}_\beta$).*

Proof. Clear from the construction. □

Lemma 15.3 (True Path Lemma)**.** $f = \liminf_s f_s$.

Proof. This is immediate from the definitions of C_α and f in §13, and f_s in Step 6A. □

Lemma 15.4. *For all $\alpha \in T$,*

(i) $f <_L \alpha \Rightarrow R_{\alpha,\infty} = \emptyset$;
(ii) $\alpha <_L f \Rightarrow Y_\alpha =^* \emptyset$;
(iii) $\alpha \subset f \Rightarrow Y_{<\alpha} =^* \emptyset$.

Proof. Given x, choose s such that $x \lhd m_s$ and $f_s <_L \alpha$. Step 6C will initialize all nodes in R_α the next time Step 6 acts, emptying the region. Now, x is γ-ineligible for all $t \geq s$ and all $\gamma \supseteq \alpha$, so x cannot be in any such $S_{\gamma,t}$. Steps 1 and 2 will not act on x and α, by construction conditions 1.2 and 2.3, so $x \notin R_{\alpha,t}$, giving (i).

For (ii), assume $\alpha <_L f$. Since by definition $|C_\alpha| < \infty$, we will only see $\alpha \subset f_s$ a finite number of times. Step 6B will act finitely-often on α and therefore there will be only a finite number of entries $\langle \alpha, \nu \rangle$ on $\mathcal{L}$. Since Step 1 marks a list entry each time it acts, only finitely-many x can enter S_α under Step 1; also, $\mathcal{L}$ can be α-marked only finitely many times, so $\lim_s m(\alpha, s) < \infty$. Step 2, by condition 2.4, will move only finitely-many x into R_α, and by Lemma 15.2, those are the only ways for x to enter R_α. Therefore $Y_\alpha =^* \emptyset$.

Part (iii) is immediate from (ii) since $<_L$ is a well-order. □

Lemma 15.5. *For every $\alpha \in T$, if $\alpha \neq \lambda$ and $\beta = \alpha^-$, then*

(i) $Y_\alpha \setminus Y_\beta = \emptyset$ *and* $Y_\alpha \subseteq Y_\beta$*;*
(ii) $(\forall x)(\exists^{\leq 1} s)[x \in R_{\alpha,s+1} - R_{\alpha,s}]$*;*
(iii) $U_\alpha \setminus Y_\alpha = \widehat{V}_\alpha \setminus Y_\alpha = \emptyset$*;*
(iv) $\alpha \subset f \Rightarrow (\exists v_\alpha)(\forall x)(\forall s \geq v_\alpha)[x \in R_{\alpha,s} \rightarrow (\forall t \geq s)[x \in R_{\alpha,t}]]$.

Proof. To see (i), note the only way for x to enter Y_α is by Step 1 or 2 moving it there, both of which require $x \in R_\beta \subseteq Y_\beta$.

For (ii), suppose $x \in R_{\alpha,s+1} - R_{\alpha,s}$ and $x \in R_{\alpha,t} - R_{\alpha,t+1}$ for some $t > s$ (i.e., it leaves again). Then by 6D, we know $x \lhd m_s$. By Lemma 15.2 (ii), at stage $t+1$ either

(1) Step 6C applies to α and x
(2) Step 1 applies to δ and x for some $\delta <_L \alpha$, $\delta = \alpha(x, t+1)$

In case (1), we know $f_{s+1} <_L \alpha$, so x is γ-ineligible for all stages $v \geq t+1$ and $\gamma \supseteq \alpha$, so x cannot re-enter R_α. In case (2), by Lemma 15.2 (ii), construction condition 1.5, and induction on $v \geq t$, there are two possibilities. The first is that for all $v \geq t$, $\alpha(x,v) <_L \alpha$ so $x \notin R_{\alpha,v}$, which happens if the only steps which apply to x are 1 and 2. The second possibility is that at some stage v, Step 6C applies to x and some $\eta <_L \alpha$ ($\eta = \alpha(x, v-1)$). In that event, we know $f_v <_L \eta <_L \alpha$, so as in case (1) $x \notin R_{\alpha,w}$ for all $w \geq v$.

Enumeration of x into U_α or $\widehat{V}_\alpha$ can take place in Step 1, 3, 4, or 5. Step 1 also puts x into Y_α, Steps 3 and 5 require $x \in S_\alpha$, and Step 4 requires $x \in R_\alpha$, so (iii) holds.

For (iv), assume $\alpha \subset f$ and choose v_α such that $\forall s \geq v_\alpha$, $f_s \not<_L \alpha$, and such that no $\beta <_L \alpha$ acts at stage s (which we can assure by Lemma 15.4 (iii)), so $Y_{<\alpha,s} = Y_{<\alpha}$. As in (ii), the only ways for x to leave R_α are by Step 1 or 6C. Step 1 would pull x to S_γ for some $\gamma <_L \alpha$, but by assumption γ is no longer acting. Step 6C would have to pull x from R_α to the left, onto the true path, but again by assumption, the true path never again appears to be to the left of α. Thus x must remain in $R_{\alpha,s}$ for all $s \geq v_\alpha$. □

Lemma 15.6. *For all x,*

(i) $\alpha(x) := \lim_s \alpha(x,s)$ *exists;*
(ii) *x is enumerated into at most finitely-many c.e. ideals U_γ, $\widehat{V}_\gamma$ such that for such an ideal X, $x \in X_{s+1}$ but $x \notin \langle X_s \rangle$ (that is, x is independent from X_s).*

Proof. For (i), if $x \in S_\alpha$, we may assume $x \rhd m_{|\alpha|}$ because both Step 1 and Step 2 require that, and they are the only ways for x to enter S_α originally. Fix x and suppose it is m_i in the enumeration of M. Let $\gamma = f \restriction i$, and let v_γ be defined as in Lemma 15.5 (iv). Choose $s > v_\gamma$

such that $\gamma \subset f_s$. Let $\delta_0 = \alpha(x, s)$. Either $\delta_0 <_L \gamma$ or $\delta_0 \subseteq \gamma$ (our choice of s prohibits $\gamma <_L \delta_0$, and $|\alpha(x, s)| < i = |\gamma|$ prevents $\gamma \subset \delta_0$). By choice of s, x can only be moved by Step 1 or 2, not by 6C. By induction on $t \geq s$, if $\delta_1 = \alpha(x, t)$ and $\delta_2 = \alpha(x, t + 1)$ are nonequal, then either $\delta_2 <_L \delta_1$ or $\delta_2 \supset \delta_1$. However, there is no infinite sequence $\{\delta_0, \ldots\}$ allowed for x such that $\forall k(\delta_{k+1} <_L \delta_k \ \vee \ \delta_{k+1} \supset \delta_k)$, because x can go no lower on the tree than level i, and $<_L$ is a well-order.

To see (ii), note that by Lemma 15.2, the only ways for independent x to be enumerated into U_γ or $\widehat{V}_\gamma$ are via Steps 1, 3, 4, and 5. Step 1 requires x be moved on the tree, and by part (i) that can only happen finitely-many times. Steps 3, 4, and 5 require that x be in a specific pocket or region, and again by part (i), x only changes pockets a finite number of times. With x at a particular location α, each of those three steps can only enumerate x into ideals U_γ, $\widehat{V}_\gamma$ for $\gamma \subseteq \alpha$, of which there are finitely-many. Therefore x is only enumerated into X such that $x \in X_{s+1}$ but $x \notin \langle X_s \rangle$ a finite number of times. ☐

Lemma 15.7. (i) *Step 6 applies infinitely often;*

(ii) *If the hypotheses of some Step 1-5 ($\hat{1}$-$\hat{5}$) remain satisfied, then that step eventually applies.*

Proof. If Step 6 applies at stage s, then Y_λ remains the same from stage s until the next time Step 6 applies; in particular, it is finite. Steps 1-5 may move balls on the tree or enumerate elements into ideals U_α or $\widehat{V}_\alpha$, where the element enumerated is not generated by the elements already in the ideal, but by Lemma 15.6 this happens only finitely many times. Therefore, Step 11 applies again at some stage $t > s$ and (i) holds.

By the design of the construction, Step 6 cannot occur if the hypotheses for some Step 1-5 ($\hat{1}$-$\hat{5}$) are satisfied, so by (i) a step whose hypotheses remain satisfied must eventually apply, and (ii) holds. ☐

Lemma 15.8. *If $\alpha \subset f$, $\alpha \neq \lambda$, and $\beta = \alpha^-$, then*

(i) $(\forall \gamma <_L f)[m(\gamma) := \lim_s m(\gamma, s) < \infty]$*;*

(ii) $m(\alpha) := \lim_s m(\alpha, s) = \infty$*;*

(iii) $\mathcal{E}_\alpha \supseteq \mathcal{M}_\alpha = \mathcal{F}^+_\beta$*;*

(iv) $\widehat{\mathcal{E}}_\alpha \supseteq \widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}^+_\beta$*;*

Proof. If $\gamma <_L f$, then $\gamma \subseteq f_s$ for only finitely-many s. Thus only finitely-many γ-entries are ever added to $\mathcal{L}$, so $\mathcal{L}$ is necessarily γ-marked only finitely often, giving (i).

To see (ii), fix $\alpha \subset f$, $\alpha \neq \lambda$, and let $\beta = \alpha^-$. By definition of f, $\alpha \subset f$ implies $\mathcal{M}_\alpha = \mathcal{F}^+_\beta$ and $\widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}^+_\beta$. Suppose $m(\alpha) < \infty$; say $m(\alpha, s) = n$ for all $s \geq s_0$.

Claim: Every α-entry $\langle\alpha,\nu_1\rangle$ on $\mathcal{L}$ ($\langle\alpha,\hat{\nu}_1\rangle$ on $\widehat{\mathcal{L}}$) is eventually marked. (Proved below.)

Using the claim, find $s > s_0$ such that $\alpha \subset f_{s+1}$ and every α-entry on $\mathcal{L}_s$ and $\widehat{\mathcal{L}}_s$ is marked. But then by Step 6B, $m(\alpha, s+1) > m(\alpha, s) = n$, which contradicts the choice of s_0.

Proof of Claim: Suppose $\langle\alpha,\nu_1\rangle \in \mathcal{L}$ is never marked. By Step 6B, then, there are only finitely-many entries on $\mathcal{L}$. Choose $s_1 \geq s_0$ such that (1) every α-entry on $\mathcal{L}$ and every entry on $\mathcal{L}$ preceding $\langle\alpha,\nu_1\rangle \in \mathcal{L}$ which will ever be marked has been marked by stage s_1; (2) $Y_{<\alpha,s_1} = Y_{<\alpha}$; and (3) for all $x \trianglelefteq m_n$, $x \in Y_{\alpha,s_1} \Leftrightarrow x \in Y_\alpha$. Such a state exists by (1) assumption, (2) Lemma 15.4 (iii), and (3) Lemma 15.6 (i). Then $Y_\alpha = Y_{\alpha,s_1}$, because no $x \vartriangleright m_n$ can enter R_α under Step 2, and no x can later enter R_α under Step 1 because it must mark an α-entry on $\mathcal{L}$. We know $\nu_1 \in \mathcal{M}_\alpha$ because $\langle\alpha,\nu_1\rangle \in \mathcal{L}$, and $\mathcal{M}_\alpha = \mathcal{F}^+_\beta$ since $\alpha \subset f$. Then, by the definition of $\mathcal{F}^+_\beta$,

$$(\exists^{np} x)(\exists s > s_1)[x \in Y_{\beta,s} \ \&\ \nu^+(\alpha, x, s) = \nu_1]$$

Almost every such x also satisfies the hypotheses of Step 1, so some such element is moved to S_α under Step 1 at some stage $s+1 > s$, which marks an entry $\langle\alpha,\nu_1\rangle$, contradicting the assumption.

The dual proof establishes the claim for $\widehat{\mathcal{L}}$. ⊣

By (ii), since $\alpha \subset f$, $\mathcal{L}$ and $\widehat{\mathcal{L}}$ are α-marked an infinite number of times. Thus for every $\nu_1 \in \mathcal{M}_\alpha$, an infinite number of entries $\langle\alpha,\nu_1\rangle$ are added to $\mathcal{L}$. Each entry is later marked by Step 1 when at some stage $s+1$, some x is moved into S_α, where x is not generated by the elements of $Y_{\alpha,s}$. Hence, $\nu_1 \in \mathcal{E}_\alpha$ and (iii) holds. Part (iv) holds by the same proof as (iii), with Step $\hat{1}$. □

Lemma 15.9. *$\alpha \subset f \Rightarrow R_{\alpha,\infty} =^\diamond Y_\alpha =^\diamond Y_\lambda = M$.*

Proof. By Lemma 15.7 (i), Step 6 applies infinitely-many times, so it must eventually put every $x \in M$ into Y_λ. By induction, assume $R_{\beta,\infty} =^\diamond Y_\beta =^\diamond M$ for $\beta = \alpha^-$. By Lemma 15.4, $Y_{<\alpha} =^* \emptyset$, and almost every $x \in R_\beta$ not yet in R_α must eventually lie in S_β. By Lemma 15.8, $m(\alpha) = \infty$ and $m(\gamma) < \infty$ for $\gamma <_L \alpha$ with $\gamma^- = \beta$. Therefore almost every $x \in R_\beta$ not yet in R_α must eventually satisfy the conditions of Step 2 of the construction and be moved to S_α. By Lemma 15.5 (iv), cofinitely many such x will remain in R_α forever, so $R_{\alpha,\infty} =^\diamond Y_\alpha =^\diamond M$. □

Lemma 15.10. *$\alpha \subset f \Rightarrow \alpha$ is $\mathcal{M}$-consistent.*

Proof. Let $\alpha \subset f$ such that $\beta = \alpha^-$ and α is not $\mathcal{M}$-consistent. That is, $e_\alpha > e_\beta$, and for some $\nu_0 \in \mathcal{M}_\alpha$, there is $\nu_1 >_B \nu_0$ such that $\nu_1 \restriction \beta \in \mathcal{M}_\beta$ but $\nu_1 \notin \mathcal{M}_\alpha$. By the definition of T, α is a terminal node, so $R_\alpha = S_\alpha$. Since $\alpha \subset f$, Lemma 15.9 says $S_{\alpha,\infty} =^\diamond M$ and Lemma 15.5 (v) gives a stage v_α such that no $x \in S_{\alpha,s}$ $(s > v_\alpha)$ ever leaves S_α. By Lemma 15.8, $\mathcal{E}_\alpha \supseteq \mathcal{M}_\alpha$, so

$$(\exists^{np} x)(\exists s)[x \in S_{\alpha,s+1} - S_{\alpha,s} \;\&\; \nu(\alpha, x, s+1) = \nu_0]. \tag{15.1}$$

Choose any such x and $s > v_\alpha$. Step 1 cannot move x, as it would cause x to leave R_α, and neither can Step 2, as there is no γ with $\gamma^- = \alpha$. Therefore Step 3 has the first chance to act on any such x. Almost every x satisfying (15.1) meets the conditions of Step 3, so Step 3 must apply to some $x \in S_{\alpha,s+1} - S_{\alpha,s}$, $t > s$, such that $\nu(\alpha, x, t) = \nu_0$. The action of Step 3 will cause $\nu(\alpha, x, t+1) = \nu_1$, with the result that α is provably incorrect for all stages $v \geq t+1$, so $\alpha \not\subset f$. □

Lemma 15.11. *If $\alpha \subset f$, then*

(i) $\widehat{\mathcal{M}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{M}\}$;
(ii) $\mathcal{M}_\alpha = \mathcal{F}_\alpha = \mathcal{E}_\alpha$;
(iii) $\widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}_\alpha = \widehat{\mathcal{E}}_\alpha$.

Proof. Part (i) is true by definition of $\widehat{\mathcal{M}}$. For (ii) and (iii), fix $\alpha \subset f$ and let $\beta = \alpha^-$. Assume by induction that the lemma holds for β. By definition, we know $\mathcal{E}_\alpha \subseteq \mathcal{F}_\alpha$, and by Lemma 15.8, we know $\mathcal{M}_\alpha \subseteq \mathcal{E}_\alpha$, and likewise on the hatted side. Therefore it suffices to show that $\mathcal{F}_\alpha \subseteq \mathcal{M}_\alpha$ and $\widehat{\mathcal{F}}_\alpha \subseteq \widehat{\mathcal{M}}_\alpha$.

Case 1. $e_\alpha = e_\beta$ and $\hat{e}_\alpha = \hat{e}_\beta$.

In this case $\mathcal{M}_\alpha = \mathcal{M}_\beta$, and since $Y_\alpha \subseteq Y_\beta$, we know $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$. By induction, $\mathcal{M}_\beta = \mathcal{F}_\beta$, so $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta = \mathcal{M}_\beta = \mathcal{M}_\alpha$.

Before Cases 2 and 3, we need a technical sublemma.

Technical Sublemma: If $e_\alpha > e_\beta$, $\nu_2 \in \langle \alpha, \sigma_2, \tau_2 \rangle \in \mathcal{F}^+_\beta$, and $\nu_1 = \langle \alpha, \sigma_1, \tau_2 \rangle$, where $\sigma_1 = \sigma_2 - \{e_\alpha\}$, then $\nu_1 \in \mathcal{F}^+_\beta$ also.

Proof. Suppose $\nu_2 \in \mathcal{F}^+_\beta$. Then $\nu_3 = \nu_2 \restriction \beta \in \mathcal{F}^+_\beta$ also, and $\mathcal{F}_\beta = \mathcal{E}_\beta$ by the inductive hypothesis. Therefore,

$$(\exists^{np} x)(\exists s)[x \in Y_{\beta,s} - Y_{\alpha,s-1} \;\&\; \nu(\beta, x, s) = \nu_3].$$

But for each such x and s, $x \notin Z_{e_\alpha,s} = \{x : x \in U_{e_\alpha,s} \;\&\; x \in Y_{\beta,s-1}\}$. Therefore $\nu^+(\alpha, x, s) = \nu_1$, and so a nonprincipal collection of x have ν^+ state ν_1 and $\nu_1 \in \mathcal{F}^+_\beta$. The dual proof shows the sublemma holds for $\mathcal{F}^+_\beta$. ⊣

Case 2. $e_\alpha > e_\beta$.

Part (ii) may be proved directly. Suppose $\nu_1 \in \mathcal{F}_\alpha$, and let $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$. Then

$$(15.2) \qquad (\exists^{np} x)(\exists s)[x \in Y_{\alpha,s} \ \&\ \nu(\alpha, x, s) = \nu_1]$$

by definition of $\mathcal{F}_\alpha$. Note that $Y_{\alpha,s} \subseteq Y_{\beta,s}$ and $\nu(\alpha, x, s) \leq_R \nu^+(\alpha, x, s)$ because $U_{\alpha,s} \subseteq Z_{e_\alpha,s}$. Suppose

$$(15.3) \qquad (\exists^{np} x)(\exists s)[x \in Y_{\alpha,s} \ \&\ \nu^+(\alpha, x, s) = \nu_1].$$

Then by the definition of $\mathcal{F}^+_\beta$, $\nu_1 \in \mathcal{F}^+_\beta$ since $Y_\alpha \subseteq Y_\beta$ and $\alpha \subset f$ gives $\mathcal{F}^+_\beta = \mathcal{M}_\alpha$.

If (15.3) fails, then for almost every x in (15.2), $\nu^+(\alpha, x, s) = \nu_2 >_R \nu_1$, so $\nu_2 = \langle \alpha, \sigma_2, \tau_1 \rangle$ where $e_\alpha \notin \sigma_1$ and $\sigma_2 = \sigma_1 \cup \{e_\alpha\}$. Again, by definition $\nu_2 \in \mathcal{F}^+_\beta$, so by the sublemma, $\nu_1 \in \mathcal{F}^+_\beta = \mathcal{M}_\alpha$.

Part (iii) is proved using the following three claims.

Claim 1. $\widehat{\mathcal{E}}_\alpha \subseteq \widehat{\mathcal{M}}_\alpha$.

Claim 2. If $\hat{x} \in \hat{Y}_{\alpha,s}$, $\nu_1 = \nu(\alpha, \hat{x}, s) \in \widehat{\mathcal{M}}_\alpha$, $s > v_\alpha$ (where v_α is defined as in Lemma 15.5 (iv)), and RED causes enumeration of $\hat{x}$ so that $\hat{\nu}_2 = \nu(\alpha, \hat{x}, s+1)$, then $\hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha$.

Claim 3. If $\hat{x} \in \hat{Y}_{\alpha,s}$, $\nu_1 = \nu(\alpha, \hat{x}, s) \in \widehat{\mathcal{M}}_\alpha$, $s > v_\alpha$ (where v_α is defined as in Lemma 15.5 (iv)), and BLUE causes enumeration of $\hat{x}$ so that $\hat{\nu}_2 = \nu(\alpha, \hat{x}, s+1)$, then $\hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha$.

Claim 1 says that the states well-visited by elements when they first enter R_α are in $\mathcal{M}_\alpha$. Claims 2 and 3 together say that after stage v_α, every state attained by an element after it is already in R_α is also in $\mathcal{M}_\alpha$, so in particular, the well-visited states are in $\mathcal{M}_\alpha$. These three suffice to show $\widehat{\mathcal{F}}_\alpha \subseteq \widehat{\mathcal{M}}_\alpha$.

Proof of Claim 1. Suppose $\hat{\nu}_1 \in \widehat{\mathcal{E}}_\alpha$. Then

$$(\exists^{np} \hat{x})(\exists s)[\hat{x} \in \hat{S}_{\alpha,s} - \hat{Y}_{\alpha,s-1} \ \&\ \nu(\alpha, \hat{x}, s) = \hat{\nu}_1]$$

For every such $\hat{x}$ and s, $\hat{x}$ must have entered $\hat{S}_{\alpha,s}$ under Step $\hat{1}$ or Step $\hat{2}$. If it was via Step $\hat{1}$, we must have marked an entry $\langle \alpha, \hat{\nu}_1 \rangle$ on $\widehat{\mathcal{L}}$, so $\hat{\nu}_1 \in \widehat{\mathcal{M}}_\alpha$ by the definition of $\widehat{\mathcal{L}}$. If Step $\hat{2}$ acted we know $\hat{x} \notin \widehat{U}_{\alpha,s}$ because Lemma 15.5 (iv) gives $\widehat{U}_\alpha \setminus \hat{Y}_\alpha = \emptyset$, so $\hat{x} \notin \hat{Y}_{\alpha,s-1} \Rightarrow \hat{x} \notin \widehat{U}_{\alpha,s-1}$ and Step $\hat{2}$ does not enumerate. Thus by (11.1), $e_\alpha \notin \sigma_1$, where $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$. Let $\nu_3 = \nu_1 \restriction \beta$. By the inductive hypothesis, $\hat{\nu}_3 \in \widehat{\mathcal{F}}_\beta = \widehat{\mathcal{M}}_\beta$, so $\nu_3 \in \mathcal{M}_\beta = \mathcal{F}_\beta$. The set $\mathcal{F}^+_\beta$ must contain a state extending ν_3, so either $\nu_1 \in \mathcal{F}^+_\beta$ or $\nu_2 \in \mathcal{F}^+_\beta$, where $\nu_2 = \langle \alpha, \sigma_1 \cup \{e_\alpha\}, \tau_1 \rangle$. If $\nu_2 \in \mathcal{F}^+_\beta$, then $\nu_1 \in \mathcal{F}^+_\beta$ by the sublemma. If not, $\nu_1 \in \mathcal{F}^+_\beta = \mathcal{M}_\alpha$ anyway, so $\hat{\nu}_1 \in \widehat{\mathcal{M}}_\alpha$. ⊣

Proof of Claim 2. Suppose RED causes enumeration of $\hat{x}$ such that $\hat{\nu}_2 = \nu(\alpha, \hat{x}, s+1)$, where $\hat{x} \in \hat{Y}_{\alpha,s}$, $\nu_1 = \nu(\alpha, \hat{x}, s) \in \widehat{\mathcal{M}}_\alpha$, and $s > v_\alpha$ (where v_α is defined as in Lemma 15.5 (iv)). Then $\hat{\nu}_1 <_R \hat{\nu}_2$, so $\nu_1 <_B \nu_2$. Since $\hat{\nu}_1 \in \widehat{\mathcal{M}}_\alpha$, $\nu_1 \in \mathcal{M}_\alpha$. Since $\alpha \subset f$, so α is $\mathcal{M}$-consistent, $\nu_2 \in \mathcal{M}_\alpha$, and thus $\hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha$. ⊣

Proof of Claim 3. Suppose BLUE causes enumeration of $\hat{x}$ such that $\hat{\nu}_2 = \nu(\alpha, \hat{x}, s+1)$, where $\hat{x} \in \hat{Y}_{\alpha,s}$, $\nu_1 = \nu(\alpha, \hat{x}, s) \in \widehat{\mathcal{M}}_\alpha$, and $s > v_\alpha$ (where v_α is defined as in Lemma 15.5 (iv)). Since $s > v_\alpha$, $\hat{x} \in \hat{R}_{\alpha,s} \cap \hat{R}_{\alpha,s+1}$. Since BLUE is the player acting, the enumeration must take place via Step $\hat{1}$, $\hat{3}$, or $\hat{5}$ applying to $\hat{x}$ and some $\gamma \supseteq \alpha$. If Step $\hat{1}$ applies, it will give $\hat{x}$ some γ-state $\hat{\nu}_3 = \hat{\nu}(\gamma, \hat{x}, s+1)$. By construction $\hat{\nu}_3 \in \widehat{\mathcal{M}}_\gamma$, so $\hat{\nu}_3 \upharpoonright \alpha = \hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha$. The same holds in Step $\hat{5}$, where for case 1, $\hat{x} \in \hat{Y}_{\gamma,s}$, and for case 2, $\hat{x} \in \hat{Y}_{\delta,s}$ for some $\delta^- = \gamma$. If the BLUE enumeration takes place in Step $\hat{3}$, γ is $\mathcal{M}$-inconsistent, so it must be that $\gamma \supsetneq \alpha$ since $\alpha \subset f$. Let $\hat{\nu}_3 = \nu(\gamma^-, \hat{x}, s+1)$. By construction condition $\hat{3}.4$, $\hat{\nu}_3 \in \widehat{\mathcal{M}}_{\gamma^-}$, so $\hat{\nu}_2 = \hat{\nu}_3 \upharpoonright \alpha \in \widehat{\mathcal{M}}_\alpha$ by the definition of T. ⊣

Case 3. $\hat{e}_\alpha > \hat{e}_\beta$.

Holds by the dual proof to Case 2. □

Lemma 15.12. *$\alpha \subset f \Rightarrow \alpha$ is $\mathcal{R}$-consistent.*

Proof. For a contradiction, assume $\alpha \subset f$ is not $\mathcal{R}$-consistent, so $(\exists \nu_1 \in \mathcal{R}_\alpha)(\forall \nu_2 \in \mathcal{M}_\alpha)[\nu_1 \not<_R \nu_2]$. Choose such a ν_1. As in Lemma 15.10, $S_\alpha = R_\alpha$, $S_\alpha =^\diamond M$, and there is v_α such that for $s > v_\alpha$, no $x \in S_{\alpha,s}$ later leaves S_α. Lemma 15.11 gives that $\mathcal{M}_\alpha = \mathcal{E}_\alpha$. By definition, $\mathcal{R}_\alpha \subseteq \mathcal{M}_\alpha$, so $\nu_1 \in \mathcal{R}_\alpha \Rightarrow \nu_1 \in \mathcal{E}_\alpha$, giving

$$(\exists^{np} x)(\exists s > v_\alpha)[x \in S_{\alpha,s+1} - Y_{\alpha,s} \ \&\ \nu(\alpha, x, s) = \nu_1]. \tag{15.4}$$

For each such x and s, as in Lemma 15.10, neither Step 1 nor Step 2 can apply at any stage $t > s+1$. Step 3 cannot apply to $x \in S_{\alpha,t}$ because, by Lemma 15.10, α is $\mathcal{M}$-consistent. Step 5 cannot apply to x while $\nu(\alpha, x, t) = \nu_1$, because it requires $\nu_1 \in \mathcal{B}_\alpha$, which is disjoint from $\mathcal{R}_\alpha$. However, if $\nu(\alpha, x, t) = \nu_1$ for all $t \geq s$, then x witnesses that $F(\alpha^-, \nu_1)$ fails, and $\nu_1 \in \mathcal{R}_\alpha$ would force $\alpha \not\subset f$. Therefore at some stage $t > s$, the state of x must be changed so that $\nu(\alpha, x, t) = \nu_1$ but $\nu(\alpha, x, t+1) = \nu_2 \neq \nu_1$. The only step remaining which can do such a thing is Step 4, which will choose ν_2 such that $\nu_1 <_R \nu_2$. This must happen for all x satisfying (15.4), so choose ν_2 such that a nonprincipal collection of x are given state ν_2. Then $\nu_2 \in \mathcal{F}_\alpha$, so by Lemma 15.11, $\nu_2 \in \mathcal{M}_\alpha$, and α is $\mathcal{R}$-consistent. □

Lemma 15.13. *If* $\alpha \subset f$ *and* $\nu_1 \in \mathcal{B}_\alpha$, *then* $\{x : x \in Y_\alpha \ \&\ \nu(\alpha, x) = \nu_1\}$ *is finite.*

Proof. Fix $\alpha \subset f$ and $\nu_1 \in \mathcal{B}_\alpha$. Let v_α be as in Lemma 15.5 (v), and let $x \in R_{\alpha,s}$ for some $s > v_\alpha$. Cofinitely many of the elements $x \in Y_\alpha$ will satisfy that hypothesis. Assume that for all $t \geq s$, $\gamma = \alpha(x, t)$ (some $\gamma \supseteq \alpha$) and $\nu_1 = \nu(\alpha, x, t)$. Since $\alpha \subset f$, by the (inductive) definition of $\mathcal{B}_\alpha$, $\nu_1 \in \mathcal{B}_\alpha \Rightarrow \nu_1' \in \mathcal{B}_\gamma$ for all $\nu_1' \in \mathcal{M}_\gamma$ such that $\nu_1' \restriction \alpha = \nu_1$. Note that x's γ-state must be some such ν_1'.
Case 1. γ is $\mathcal{R}$-consistent and $\mathcal{M}$-consistent.

Then the hypotheses of Step 5 case 1 remain satisfied, so at some stage $t + 1 > s$, it applies with $\nu_1' = \nu(\gamma, x, t)$, $\nu_2' = \nu(\gamma, x, t + 1)$, $\nu_1' <_B \nu_2'$, and $\nu_2' \in \mathcal{M}_\gamma - \mathcal{B}_\gamma$. Hence $\nu_2 = \nu_2' \restriction \alpha \in \mathcal{M}_\alpha - \mathcal{B}_\alpha$ and $\nu(\alpha, x, t + 1) = \nu_2 >_B \nu_1$.
Case 2. Otherwise.

Then likewise, Step 5 case 2 applies to x and γ^-. □

Lemma 15.14. *The correspondence* $U_\alpha \leftrightarrow \widehat{U}_\alpha$ *and* $\widehat{V}_\alpha \leftrightarrow V_\alpha$, $\alpha \subset f$, *defines an isomorphism from* $G^\diamondsuit$ *to* $\mathcal{E}^*$.

Proof. Choose $\alpha \subset f$. By Lemmas 15.10 and 15.12 α is consistent. Therefore every $\alpha \subset f$ has an extension in f and f is infinite. By Lemma 15.9 and its dual, $Y_\alpha =^\diamondsuit M$ and $\hat{Y}_\alpha =^* \omega$. Lemma 15.11 gives $\mathcal{F}_\alpha = \mathcal{M}_\alpha = \widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}_\alpha$, so the well-visited states on the M and $\widehat{\omega}$ sides coincide. Since for $\alpha \subset f$, $Y_\lambda - Y_\alpha =^\diamondsuit \emptyset$ ($\hat{Y}_\lambda - \hat{Y}_\alpha =^* \emptyset$), Lemma 15.13 and its dual give $\mathcal{N}_\alpha = \widehat{\mathcal{N}}_\alpha$ (by the remarks preceding (12.12)), so the non-well-resided states also coincide. Therefore we have met the automorphism requirement stated and discussed in §10. □

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