FINITE SELF-INFORMATION

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ABSTRACT. We present a definition, due to Levin, of mutual information $I(A : B)$ for infinite sequences. We say that a set A *has finite self-information* if $I(A : A) < \infty$. It is easy to see that every K-trivial set has finite self-information. We answer a question of Levin by showing that the converse does not hold. Finally, we investigate the connections between having finite self-information and other notions of weakness such as jump-traceability. In particular, we show that our proof can be adapted to produce a set that is low for both effective Hausdorff dimension and effective packing dimension, but not K-trivial.

We think of the information content of a finite binary string σ as its prefix-free Kolmogorov complexity $K(\sigma)$. The idea of using plain Kolmogorov complexity to measure information content was proposed by Kolmogorov in [10]; for both technical and philosophical reasons plain complexity is often replaced by prefix-free Kolmogorov complexity (as defined by Levin [12] and later Chaitin [3]) in this and many other applications. The *mutual information* of two finite strings σ, τ is $K(\sigma) + K(\tau) - K(\sigma, \tau)$, where $K(\sigma, \tau) := K(\langle \sigma, \tau \rangle)$ for a standard pairing function $\langle \cdot, \cdot \rangle$. (For more on Kolmogorov complexity and related concepts, see [15].)

The proper extension of this definition from the finite to the infinite case is not clear-cut. A computable infinite binary sequence should contain finitely-much information, as coded by its generating program. One of the most important notions in the theory of algorithmic randomness is that of a K-*trivial* sequence, which is an infinite sequence A such that $K(A \restriction n) \leqslant K(n) + c$ for some constant c independent of n (see Chapter 11 of [6] for more information on K-triviality). Since we are using prefix-free Kolmogorov complexity to measure information, a K-trivial sequence should also contain finitely-much information, since

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K cannot differentiate between such a sequence and a computable sequence. Conversely, any non-K-trivial sequence should possess infinite information. Our purpose in this paper is not to propose a definition for the information content of an individual sequence, though both $\sup\{K(\sigma) - K^A(\sigma) : \sigma \in 2^{<\omega}\}$ and $\limsup_n(K(A \restriction n) - K(n))$ satisfy the conditions above, but these intuitions provide a means to judge the quality of a definition of mutual information for infinite sequences. (Here K^A is prefix-free Kolmogorov complexity relativized to A.)

Several such definitions have been proposed by Levin [12, 13, 14]. In these papers, Levin has also argued for the importance of the notion of mutual information in a wide range of applications. We focus on the definition given by Levin in [12], and used in that paper to prove theorems on information conservation. This definition has a number of desirable properties, and in many ways fulfills the intuition for the infinite case. The mutual information of A and B, denoted by $I(A : B)$ and defined in the following section, is symmetric up to a constant, and when A and B are finite strings it simplifies to $K(A)+K(B)-K(A, B)$. Furthermore, if A is K-trivial then $I(A : B)$ is finite for all B, and as proved below, if A is not K-trivial then there is some B for which $I(A : B)$ is infinite.

During the AIM Workshop in Effective Randomness (held at ARCC, Palo Alto, CA, 7–11 August 2006), which the authors attended, Levin asked whether $I(A : A) < \infty$ implies A is K-trivial. Here we answer this question negatively, constructing a non-K-trivial set with finite self-information. Separating these classes leads us to explore, in the final section of the paper, the relationships between having finite self-information and other weakness notions. In particular, we show that our construction can be modified to produce non-K-trivial sets that are low for effective dimension, as defined in that section.

At the workshop Levin also asked the "soft" question of which definition of mutual information is the "right" one. Our result might be seen as evidence against the definition presented below. However, we may argue for its plausibility in the face of our result by considering the possession of encoded information. If the second author has a cookbook in Portuguese, full of recipes unfamiliar to her, this does not help her make any new dishes. However, she possesses information, as evidenced by the fact that teaming up with the first author—who knows Portuguese and hence can decode the recipes—she can make meals she could not previously. This information simply does not contribute to her self-information. This idea of "hidden information", which a sequence may not be powerful enough to extract from itself,

but which may be extracted by another sequence of different computational power, manifests itself in the contrast between Proposition 3 and Theorem 6 below.

We will use some basic facts about Kolmogorov complexity and algorithmic randomness. As usual, we identify a set A with the infinite sequence corresponding to its characteristic function. All our logarithms are taken with base 2. Terms and notation not defined below may be found in Downey and Hirschfeldt [6]. In particular, we will use the KC Theorem (Theorem 3.6.1 in [6]): The *weight* of a sequence of pairs $\langle n_i, \sigma_i \rangle_{i \in \omega}$, where the n_i are natural numbers and the σ_i are either finite binary strings or natural numbers, is $\sum_i 2^{-n_i}$. Recall that such a sequence is a *KC set* if it is a computable enumeration and its weight is finite. The KC Theorem states that for such a set, $K(\sigma_i) \leqslant^+ n_i$ for all i. (Here and below, a superscript $+$ on a relation means the relation holds up to an additive constant.) We will also use the Recursion Theorem in conjunction with the KC Theorem; for a discussion of this technique, see Section 3.6 in [6]. Another important tool will be the result due to Nies [19] that if A is K-trivial then it is *low for K*; that is, $K^A(\sigma) =^+ K(\sigma)$ for all σ.

1. Mutual Information

Definition 1 (Levin [12])**.** The *mutual information* of infinite binary sequences A and B is

$$I(A : B) = \log \sum_{\sigma, \tau \in 2^{<\omega}} 2^{-K(\sigma, \tau) - K^A(\sigma) - K^B(\tau) + K(\sigma) + K(\tau)}.$$

Note that this definition is independent of the choice of universal prefix-free machine up to an additive constant. Levin [12] showed that if we replace A and B in the above definition by finite strings α and β, then $I(\alpha : \beta)$ is equal (up to an additive constant) to the usual definition of mutual information for finite strings, namely $K(\alpha) + K(\beta) - K(\alpha, \beta)$.

We may think of this definition as follows. One way to measure the mutual information of individuals A and B is to run through all pairs of subjects, and for each pair (σ, τ), to determine how much information A and B have about σ and τ, respectively. The weight we should give to each pair depends on the relationship between its elements; the more closely related σ and τ are, the more the pair should matter in our computation. For example, the fact that A knows about snakes and B knows about lizards should contribute more to the computation of the mutual information of A and B than the fact that A

knows about snakes and B knows about doorknobs. The degree of relatedness of strings σ and τ is represented by their mutual information $I(\sigma : \tau) = K(\sigma) + K(\tau) - K(\sigma, \tau)$; in the extreme case $\sigma = \tau$ this is essentially $K(\sigma)$, the total information in σ, and if σ and τ are mutually relatively random it is essentially 0. From this quantity we want to include only the portion corresponding to A's knowledge of σ and B's knowledge of τ. Since the relativized complexity $K^A(\sigma)$ may be thought of as measuring what A *does not* know about σ, we subtract it from $I(\sigma : \tau)$, and similarly we subtract $K^B(\tau)$. See [12] for further comments on the motivations behind this definition, and a way to think of it in terms of a priori probabilities.

The next two facts (the first of which was noted by several people at the aforementioned AIM workshop) show that this definition gives, at least to some degree, the desired split between finite information and infinite information.

Proposition 2. *If A is K-trivial then $I(A : B)$ is finite for all B.*

Proof. Let A be K-trivial and B be any set. As mentioned in the introduction, $K^A(\sigma) =^+ K(\sigma)$ for all $\sigma \in 2^{<\omega}$. Since $K(\sigma, \tau) \geqslant^+ K(\tau)$ for all σ and τ, we see that $I(A : B)$ is bounded above (up to a constant) by

$$\log \sum_{\tau \in 2^{<\omega}} 2^{-K^B(\tau)},$$

which is finite by Kraft's Inequality (see Section 3.6 of [6]). $\qquad\square$

Proposition 3 (with Jan Reimann)**.** *If A is not K-trivial, there exists B such that $I(A : B) = \infty$. In particular, any $B \geqslant_T A \oplus \emptyset'$ has infinite mutual information with A.*

Proof. Let $B \geqslant_T A \oplus \emptyset'$; note that B can compute K and approximate K^A from above. We choose a subsequence of initial segments of A and use B to compress them highly, making the mutual information sum diverge. To do so, we build a KC set M relative to B.

Since A is not K-trivial, for each n we have some $\tau_n \prec A$ such that $K(\tau_n) - K^A(\tau_n) \geqslant n$. Since B can compute K, and its approximation to $K^A(\tau)$ for any τ can only decrease, B can effectively find such a τ_n for each n. Let $M = \{\langle n, \tau_n \rangle \mid n \in \omega\}$. Then M is a KC set relative to B, and hence $K^B(\tau_n) \leqslant^+ n$ for all n. Since $K(\tau_n, \tau_n) =^+ K(\tau_n)$, we have

$$I(A : B) \geqslant \log \sum_n 2^{K(\tau_n)-K^A(\tau_n)-K^B(\tau_n)-O(1)} \geqslant \log \sum_n 2^{n-n-O(1)} = \infty.$$

$\qquad\square$

It is open whether there is an A for which $B \geqslant_{\mathrm{T}} A \oplus \emptyset'$ is actually *required* to obtain $I(A : B) = \infty$. We show in the following section that $B = A$ does not necessarily suffice, though, as noted by Levin [12], every 1-random set has infinite mutual information with itself and hence the cases for which $B = A$ does not suffice have measure zero.

One may ask whether letting the sum in Definition 1 range over all pairs $\sigma, \tau \in 2^{<\omega}$ is necessary. We may simplify to only $\sigma = \tau$, obtaining the following definition.

Definition 4 (Levin [12])**.** The *simplified mutual information* of infinite binary sequences A and B is

$$I'(A : B) = \log \sum_{\sigma \in 2^{<\omega}} 2^{-K^A(\sigma) - K^B(\sigma) + K(\sigma)}.$$

It is clear that $I'(A : B) = \infty$ implies $I(A : B) = \infty$. However, it is open whether the converse implication holds, both in general and for the special case of $A = B$, where the answer may differ. (Proposition 3 does hold for I' in place of I, with essentially the same proof.) A priori, what $I(A : B)$ accounts for that $I'(A : B)$ does not, at least not obviously, is information about related subjects. If A knows about snakes and B about lizards, these are sufficiently related topics that A and B share significant information. However, there does exist a single "snakes and lizards" string about which they should both know, and hence the plausibility of the idea that $I'(A : B) = \infty \iff I(A : B) = \infty$.

A related question also comes from [12]. As mentioned above, for finite strings σ, τ, we have $I(\sigma : \tau) =^+ K(\sigma) + K(\tau) - K(\sigma, \tau)$. The corresponding equality for $I'(\sigma : \tau)$ appears to be open.

2. Finite Self-Information

As mentioned above, $I(A : A) < \infty$ for all K-trivial A, and $I(A : A) = \infty$ for all 1-random A. We make the following definition.

Definition 5. A set A *has finite self-information* if $I(A : A) < \infty$.

Our main result, which we prove in this section, shows that this class of sets *properly* contains the K-trivials, answering the aforementioned question of Levin.

Theorem 6. *There is a non-K-trivial set A that has finite self-information.*

Relationships between the class of sets that have finite self-information and other classes defined by weakness notions are explored in the next section. The structure of this class is largely unknown, and the mutual information of sets that have finite self-information with sets other than themselves may be of interest. For example, perhaps a set A that has finite self-information but is not K-trivial requires $B \geqslant_{\mathrm{T}} A \oplus \emptyset'$ to have $I(A : B) = \infty$. Less drastically, perhaps if both A and B have finite self-information then $I(A : B) < \infty$ always holds.

We need a particular function for the proof of Theorem 6.

Lemma 7. *There is a function f such that $\sum_{\sigma,\tau} 2^{-K(\sigma,\tau)+f(\sigma)+f(\tau)}$ converges and f has a computable approximation for which*

$$(1) \qquad\qquad (\forall i)(\forall^\infty \sigma)(\forall s)[f_s(\sigma) > i].$$

Before proving this lemma, we note its connection with finite self-information.

Lemma 8. *If $\sum_{\sigma,\tau} 2^{-K(\sigma,\tau)+f(\sigma)+f(\tau)}$ converges and*

$$(2) \qquad\qquad K(\sigma) \leqslant^+ K^A(\sigma) + f(\sigma)$$

for all $\sigma \in 2^{<\omega}$, then A has finite self-information.

Proof. For such an A, we have

$$I(A : A) = \log \sum_{\sigma,\tau} 2^{-K(\sigma,\tau)-K^A(\sigma)-K^A(\tau)+K(\sigma)+K(\tau)}$$

$$\leqslant \log \sum_{\sigma,\tau} 2^{-K(\sigma,\tau)+f(\sigma)+f(\tau)+O(1)} < \infty.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

To prove Lemma 7, we use two auxiliary functions, given by the following lemmas.

Lemma 9. *There is a function h such that $\sum_\sigma 2^{-K(\sigma)+2h(\sigma)}$ converges and h has a computable approximation for which*

$$(3) \qquad\qquad (\forall i)(\forall^\infty \sigma)(\forall s)[h_s(\sigma) > i].$$

Note that if we were working with the simplified mutual information of Definition 4, Lemma 8 would hold with this function h in place of f.

Proof of Lemma 9. Noneffectively, h is defined as follows. Fix an approximation Ω_s to Ω. (Here Ω is the halting probability of a universal prefix-free machine; see [6] for more on this number.) We assume this

approximation is given by a universal prefix-free machine U subject to the usual convention that if $U_s(\tau)\!\downarrow = \sigma$, then $s > |\sigma|$. Let $s_0, s_1, \ldots$ be the nondecreasing sequence such that $s_0 = 0$ and s_{i+1} is the first stage at which our approximation is within 8^{-i} of Ω. Given σ, let t be the first stage at which $K_t(\sigma) = K(\sigma)$. For the unique i such that $s_i < t \leqslant s_{i+1}$, let $h(\sigma) = i$.

If $h(\sigma) = i$, then there is a τ such that $|\tau| = K(\sigma)$ and $U(\tau)\!\downarrow = \sigma$ but $U_{s_i}(\tau)\!\uparrow$. Thus, $\sum_{\sigma:h(\sigma)=i} 2^{-K(\sigma)} \leqslant \Omega - \Omega_{s_i}$. If $i > 0$, then this quantity is at most 8^{-i}, so

$$\sum_\sigma 2^{-K(\sigma)+2h(\sigma)} \leqslant \sum_{\sigma:h(\sigma)=0} 2^{-K(\sigma)} + \sum_{i>0} \sum_{\sigma:h(\sigma)=i} 2^{2i} 2^{-K(\sigma)}$$

$$\leqslant \Omega + \sum_{i>0} 4^i 8^{-i} = \Omega + \sum_{i>0} 2^{-i} < \infty.$$

To approximate h, we first approximate the sequence (s_i). Let $s_0[t] = 0$ for all t. For $i > 0$ and t, let $s_i[t]$ be the least s such that $\Omega_t - \Omega_s \leqslant 8^{-i}$. Clearly, $s_i[t] \leqslant s_i[t+1]$ for all i and t, and $\lim_t s_i[t] = s_i$ for all i.

To define $h_t(\sigma)$, let $t' \leqslant t$ be the most recent $t' > |\sigma|$ at which $K_{t'}(\sigma) < K_{t'-1}(\sigma)$, or $t' = |\sigma| + 1$ if there is none. Let i be such that $s_i[t] < t' \leqslant s_{i+1}[t]$, and set $h_t(\sigma) = i$.

It is clear that this approximation is computable and has limit h. If $|\sigma| \geqslant s_i$ then $|\sigma| \geqslant s_i[t]$ for all t, so $h_t(\sigma) \geqslant i$ for all t. Thus, our approximation to h satisfies (3). $\qquad\square$

Lemma 10. *There are a 2-place function g and a constant r such that for every σ we have $\sum_\tau 2^{-K(\sigma,\tau)+g(\sigma,\tau)} < 2^{-K(\sigma)+r}$, and furthermore g has a computable approximation such that for each σ,*

$$(4) \qquad\qquad (\forall i)(\forall^\infty \tau)(\forall s)[g_s(\sigma,\tau) > i].$$

Proof. The proof of this lemma is similar to that of Lemma 9. Indeed, it would not be difficult to obtain Lemma 9 as a corollary to this lemma, but we chose not do so for clarity of exposition.

Let Ω^σ be the sum of $2^{-|\rho|}$ over all ρ that are descriptions of (σ,τ) for some τ. The function $\sigma \mapsto -\log \Omega^\sigma$ is a complexity measure, so by the minimality of K among such measures, there is a constant r such that $2^{-K(\sigma)+r} > \Omega^\sigma$ for all σ. (Here $\log r$ means the base 2 logarithm of r, rounded up to the nearest larger integer. See Section 3.7 of [6] for more details on the minimality of K among complexity measures.) We can choose r so that $2^{-K(\sigma)+r} - \Omega^\sigma > 1$.

Noneffectively, g is defined as follows. Fix uniform approximations Ω_s^σ to Ω^σ for each σ, with the same assumption as in the proof of Lemma

9. For a given σ, let $s_0, s_1, \ldots$ be the nondecreasing sequence such that $s_0 = 0$ and s_{i+1} is the first stage at which our approximation is within 4^{-i} of Ω^σ. Given σ, let t be the first stage at which $K_t(\sigma, \tau) = K(\sigma, \tau)$. For the unique i such that $s_i < t \leqslant s_{i+1}$, let $g(\sigma, \tau) = i$.

If $g(\sigma, \tau) = i$, then there is a ρ such that $|\rho| = K(\sigma, \tau)$ and $U(\rho)\!\downarrow = (\sigma, \tau)$ but $U_{s_i}(\rho)\!\uparrow$, where U is our fixed universal prefix-free machine. Thus, for each σ, we have $\sum_{\tau:g(\sigma,\tau)=i} 2^{-K(\sigma,\tau)} \leqslant \Omega^\sigma - \Omega^\sigma_{s_i}$. If $i > 0$, then this quantity is at most 4^{-i}, so

$$\sum_\tau 2^{-K(\sigma,\tau)+g(\sigma,\tau)} \leqslant \sum_{\tau:h(\sigma,\tau)=0} 2^{-K(\sigma,\tau)} + \sum_{i>0} \sum_{\tau:g(\sigma,\tau)=i} 2^i 2^{-K(\sigma,\tau)}$$

$$\leqslant \Omega^\sigma + \sum_{i>0} 2^i 4^{-i} = \Omega^\sigma + \sum_{i>0} 2^{-i} = \Omega^\sigma + 1 < 2^{-K(\sigma)+r}.$$

To approximate g, given σ, we first approximate the sequence (s_i). Let $s_0[t] = 0$ for all t. For $i > 0$ and t, let $s_i[t]$ be the least s such that $\Omega^\sigma_t - \Omega^\sigma_s \leqslant 4^{-i}$. Clearly, $s_i[t] \leqslant s_i[t+1]$ for all i and t, and $\lim_t s_i[t] = s_i$ for all i.

To define $g_t(\sigma, \tau)$, let $t' \leqslant t$ be the most recent $t' > |\tau|$ at which $K_{t'}(\sigma, \tau) < K_{t'-1}(\sigma, \tau)$, or $t' = |\tau| + 1$ if there is none. Let i be such that $s_i[t] < t' \leqslant s_{i+1}[t]$, and set $g_t(\sigma, \tau) = i$.

It is clear that this approximation is computable and has limit g. For a fixed σ and s_i as above, if $|\tau| \geqslant s_i$ then $|\tau| \geqslant s_i[t]$ for all t, so $g_t(\sigma, \tau) \geqslant i$ for all t. Thus, our approximation to g satisfies (4). $\square$

Proof of Lemma 7. Let h be as in Lemma 9 and g as in Lemma 10. We may assume that $g(\sigma, \sigma) = 0$ for all σ. Let $<_l$ be the length-lexicographic ordering on strings. Define

$$f(\tau) = \min_{\sigma \leqslant_l \tau}(h(\sigma) + g(\sigma, \tau)).$$

Letting r be as in Lemma 10, we have

$$\sum_{\sigma,\tau} 2^{-K(\sigma,\tau)+f(\sigma)+f(\tau)} \leqslant 2 \sum_\sigma \sum_{\tau \geqslant_l \sigma} 2^{-K(\sigma,\tau)+f(\sigma)+f(\tau)}$$

$$= 2 \sum_\sigma 2^{f(\sigma)} \sum_{\tau \geqslant_l \sigma} 2^{-K(\sigma,\tau)+f(\tau)} \leqslant 2 \sum_\sigma 2^{h(\sigma)} \sum_{\tau \geqslant_l \sigma} 2^{-K(\sigma,\tau)+h(\sigma)+g(\sigma,\tau)}$$

$$\leqslant 2 \sum_\sigma 2^{2h(\sigma)} 2^{-K(\sigma)+r} = 2^{r+1} \sum_\sigma 2^{-K(\sigma)+2h(\sigma)},$$

which converges by the choice of h.

Let $f_s(\tau) = \min_{\sigma \leqslant_l \tau}(h_s(\sigma) + g_s(\sigma, \tau))$, where h_s and g_s satisfy (3) and (4), respectively. For a given i, there are only finitely many σ such that $h_s(\sigma)$ is ever less than i. For each such σ, there are only finitely

many τ such that $g_s(\sigma, \tau)$ is ever less than i. Thus, for all but finitely many τ, we have $h_s(\sigma) + g_s(\sigma, \tau) \geqslant i$ for all σ and s. Hence, f_s has property (1). $\qquad\square$

We assume without loss of generality that the function f from Lemma 7 is approximated from above; that is, that $f_{s+1}(\sigma) \leqslant f_s(\sigma)$ for all s and σ. If it is not originally, then we may replace $f(\sigma)$ with $\min_t f_t(\sigma)$ and $f_s(\sigma)$ with $\min_{t \leqslant s} f_t(\sigma)$, and the desired properties will still hold.

Before proceeding to the proof of Theorem 6, we establish a couple of useful lemmas.

Lemma 11. *By sufficiently speeding up our enumeration of stages, we may assume that there is a b such that for each n, c, and v, the number of strings τ of length n for which there is an s with $K_s(\tau) \leqslant K_s(n) + c = v$ is bounded above by 2^{c+b}.*

Proof. By Chaitin's Counting Lemma (Theorem 3.7.6 in [6]), there exists a constant b such that for every n there are fewer than 2^{c+b}-many τ of length n with $K(\tau) \leqslant K(n) + c$, so we may assume the approximation to K is sped up so that for each s we have fewer than 2^{c+b}-many τ of length n with $K_s(\tau) \leqslant K_s(n) + c$. If $t > s$ and $K_t(n) = K_s(n)$, then $K_s(\tau) \leqslant K_s(n) + c$ implies that $K_t(\tau) \leqslant K_t(n) + c$, so for each v, there are fewer than 2^{c+b}-many τ of length n for which there is an s with $K_s(\tau) \leqslant K_s(n) + c = v$. $\qquad\square$

Define an *oracle KC set* to be a computable enumeration of triples $\langle n_i, \sigma_i, \tau_i \rangle$ such that for the finitely-many triples listed at stage s (if any), we have $|\tau_i| = s$, and there is a p such that for any $X \in 2^\omega$, we have $\sum_{\tau_i \prec X} 2^{-n_i} \leqslant p$. The proof of the following result is a straightforward modification of the proof of the KC Theorem, as in Section 3.6 of [6].

Lemma 12. *We can speed up our enumeration of stages sufficiently to ensure that for each oracle KC set $\langle n_i, \sigma_i, \tau_i \rangle_{i \in \omega}$, there is a t such that if $\langle n_i, \sigma_i, \tau_i \rangle$ is listed at stage $s > t$ then $K_{s+1}^{\tau_i}(\sigma_i) \leqslant^+ n_i$.*

We are now ready to prove our main result.

Proof of Theorem 6. We assume in our construction that we have sped up our enumeration of stages so that Lemmas 11 and 12 apply. We also assume the usual convention that the use of any computation at stage s is at most s. By A_s we mean what we have enumerated into

A by the beginning of stage s. For the largest number r seen in the construction by the beginning of stage s, we identify A_s with the finite string $A_s \restriction r + 1$.

To make A have finite self-information, we use Lemma 8. We ensure (2) in the usual way, building a KC set M by enumerating axioms $\langle K_s^{A_s}(\sigma) + f_s(\sigma), \sigma \rangle$ whenever needed.

To make A non-K-trivial, for each constant c we define some n_c such that either $K(A \restriction n_c) > K(n_c) + c$ or $K^A(n_c) < K(n_c) - c$. (If the first disjunct holds for infinitely-many c, then A is not K-trivial; if the second one holds for infinitely-many c, then A is not low for K, and hence is again not K-trivial.)

For each c, we have a strategy for defining such an n_c. This will be a finite injury priority construction; as usual, the c strategy has priority over the $\hat{c}$ strategy if $c < \hat{c}$. The module for a single c is as follows. Let $C(x, s)$ be the set of all strings σ such that there exists a minimal-length A_s-description of σ at stage s with use $> x$. These are the strings σ for which the approximation to $K^A(\sigma)$ might go up after stage s if we put x into A after that stage.

We will enumerate numbers $m_{c,i}$ and $n_{c,i}$ for i in some initial segment of ω. By the KC Theorem, there is a d (independent of c) such that $K(n_{c,i}) \leqslant c + i + d$ for all i for which $n_{c,i}$ is defined. By the Recursion Theorem, we may assume that we know d. We may also assume that we have sped our enumeration of stages sufficiently so that $K_s(n_{c,i}) \leqslant c + i + d$ for all s such that $n_{c,i}$ is defined.

Begin with $i = 0$. Let b be as in Lemma 11. Pick fresh, large numbers $m_{c,i}$, $n_{c,i}$ such that the interval $(m_{c,i}, n_{c,i})$ contains $(c + i + d)2^{c+b}$-many numbers. (As we will show below, the size of this interval is an upper bound on how often we may want to change A to ensure that $K_s(A_s \restriction n_{c,i}) > K_s(n_{c,i}) + c$. There will be at most 2^{c+b} many changes needed for each new value of $K_s(n_{c,i})$.)

There are three actions we might take on behalf of c. The first two ensure that either A is not low for K with constant c or A is not K-trivial with constant c. The third will be needed in showing that M is indeed a KC set. It ensures that if we enumerate a number into A for the sake of c, and this enumeration causes $\langle K_s^{A_s}(\sigma) + f_s(\sigma), \sigma \rangle$ to enter M, then $f_s(\sigma)$ is not too small. (As we will show below, requiring that $f_s(\sigma) \geqslant 4c$ suffices.)

First, whenever we see

$$(5) \qquad\qquad K_{s+1}^{A_s}(n_{c,i}) < K_{s+1}(n_{c,i}) - c,$$

we suspend the action of the c strategy and restrain A up to the use of the computation witnessing the current value of $K_{s+1}^{A_s}(n_{c,i})$ by initializing all weaker-priority requirements. (The reason we use K_{s+1} here instead of K_s will be clear when we apply Lemma 12 below.) The suspension continues until, if ever, (5) no longer holds. If the strategy remains suspended forever, then $n_{c,i}$ witnesses that A is not low for K with constant c.

Second, if action on behalf of c is not suspended, whenever we see $K_s(A_s \upharpoonright n_{c,i}) \leqslant K_s(n_{c,i}) + c$ we enumerate a new number from $(m_{c,i}, n_{c,i})$ into A. We will show that we never run out of space for these enumerations, and hence this action ensures that if the c strategy is not eventually suspended forever, then $K(A \upharpoonright n_{c,i}) > K(n_{c,i}) + c$, and hence A is not K-trivial with constant c.

Third, if we ever see $f_s(\sigma) < 4c$ for any $\sigma \in C(m_{c,i}, s)$, we initialize the c strategy by increasing i by one and picking fresh, large $m_{c,i}$, $n_{c,i}$ such that again the interval $(m_{c,i}, n_{c,i})$ contains $(c + i + d)2^{c+b}$-many numbers. We also initialize weaker-priority strategies, which restrains $A \upharpoonright m_{c,i}$ for the new value of $m_{c,i}$, to attempt to ensure that the $\sigma \in C(m_{c,i}, s)$ have minimal-length A-descriptions with use $> m_{c,i}$. This action ensures that we eventually find an i such that $f(\sigma) \geqslant 4c$ for all $\sigma \in C(m_{c,i}, s)$, since it ensures that each σ can be in only finitely-many $C(m_{c,i}, s)$.

We now proceed with the full construction and its verification.

Construction. Let b and d be as above. Each strategy has a counter (corresponding to the number i above), which is initially set to 0. Initially no $m_{c,i}$ or $n_{c,i}$ is defined, and A and M are both empty. At stage s, act as follows.

First, for any σ such that $K_s^{A_s}(\sigma) + f_s(\sigma) < K_t^{A_t}(\sigma) + f_t(\sigma)$ for all $t < s$ (where we take $K_t^{A_t}(\sigma) + f_t(\sigma) = \infty$ if $t \leqslant |\sigma|$), enumerate $\langle K_s^{A_s}(\sigma) + f_s(\sigma), \sigma \rangle$ into M.

Next, proceed as follows for each c. Let i be the current value of the counter associated with c. If the c strategy was suspended at the end of stage $s - 1$ but $K_{s+1}(n_{c,i}) < K_s(n_{c,i})$, then declare the c strategy not to be suspended. If the c strategy is not currently suspended and $K_{s+1}^{A_s}(n_{c,i}) < K_{s+1}(n_{c,i}) - c$, then declare the c strategy to be suspended. We say that this strategy is newly suspended at stage s.

Finally, act for the least c that requires attention, where the attention that may be required and the action to be taken are as follows. Here i is the current value of the counter associated with c. When we say

that we initialize a strategy, we mean that we increase its counter and declare it not to be suspended.

(i) The numbers $m_{c,i}, n_{c,i}$ are not defined. In this case, define $m_{c,i}$ and $n_{c,i}$ to be fresh, large numbers such that the interval $(m_{c,i}, n_{c,i})$ contains $(c + i + d)2^{c+b}$-many numbers.

(ii) The c strategy is newly suspended at s. In this case, initialize all weaker-priority strategies.

(iii) The c strategy is not suspended, the numbers $m_{c,i}, n_{c,i}$ are defined, and there is a $\sigma \in C(m_{c,i}, s)$ with $f_s(\sigma) < 4c$. In this case, initialize the c strategy and all weaker-priority strategies.

(iv) The c strategy is not suspended, the numbers $m_{c,i}, n_{c,i}$ are defined, and $K_s(A_s \restriction n_{c,i}) \leqslant K_s(n_{c,i}) + c$. In this case, enumerate a new number from $(m_{c,i}, n_{c,i})$ into A (we will show below that there always is such a number not yet in A), and initialize all weaker-priority strategies.

Verification. The following lemma shows that A is not K-trivial, as it implies that there are either infinitely many c such that A is not low for K with constant c, or infinitely many c such that A is not K-trivial with constant c.

Lemma 13. *For each c, the c strategy acts only finitely often, and its counter has a final value i. Either the c strategy is eventually suspended forever, in which case $K^A(n_{c,i}) < K(n_{c,i}) - c$, or it ensures that $K(A \restriction n_{c,i}) > K(n_{c,i}) + c$.*

Proof. Fix c and suppose by induction that all stronger-priority strategies have ceased acting by some stage s^*. The c strategy is initialized at a stage $s \geqslant s^*$ only when there is a $\sigma \in C(m_{c,i}, s)$ with $f_s(\sigma) < 4c$, where i is the value of its counter at stage s. Let us say that such a σ causes this initialization. It is easy to check from the construction that the numbers put in by strategies for $\hat{c} \geqslant c$ at stages greater than or equal to s are all greater than $m_{c,i}$. Thus, by the assumption that stronger-priority strategies have stopped acting, A cannot change below $m_{c,i}$ at any stage greater than or equal to s, so the minimal-length A_s-description witnessing the fact that $\sigma \in C(m_{c,i}, s)$ is in fact an A-description. It follows that each σ can cause only finitely-many initializations of the c strategy. By property (1) of f from Lemma 7, there are only finitely-many σ such that $f_s(\sigma) < 4c$ for some s; that is, there are only finitely-many σ that can ever cause an initialization of the c strategy. Thus the c strategy is initialized only finitely often.

Let s^{**} be the last stage at which c is initialized (or 0 if there is no such stage). Let i be the value of the counter associated with c at stage

s^{**}, and note that this is the final value of this counter. If the c strategy is newly suspended at a stage $s > s^{**}$, then $K_{s+1}^{A_s}(n_{c,i}) < K_{s+1}(n_{c,i}) - c$. Since weaker-priority strategies are initialized at stage s, as long as the c strategy is suspended, A will not change below the use of the computation that gives $K_{s+1}^{A_s}(n_{c,i})$ its value. If the c strategy is permanently suspended after stage s, then $K(n_{c,i}) = K_{s+1}(n_{c,i})$, so $K^A(n_{c,i}) \leqslant K_{s+1}^{A_s}(n_{c,i}) < K(n_{c,i}) - c$. Otherwise, $K(n_{c,i}) < K_{s+1}(n_{c,i})$, which can happen for only finitely-many s. Thus either the c strategy is eventually suspended forever, in which case $K^A(n_{c,i}) < K(n_{c,i}) - c$ and the c strategy acts only finitely often, or the c strategy is suspended for only finitely-many stages.

So suppose we are in the latter case. Every time the c strategy enumerates a number in $(m_{c,i}, n_{c,i})$ into A at stage s, it is because $K_s(A_s \restriction n_{c,i}) \leqslant K_s(n_{c,i}) + c$. This enumeration ensures that $A_t \restriction n_{c,i} \neq A_s \restriction n_{c,i}$ for all $t > s$. By our choice of b and the assumption that $K_t(n_{c,i}) \leqslant c + i + d$ for all t, there are no more than $(c+i+d)2^{c+b}$-many τ of length $n_{c,i}$ with $K_s(\tau) \leqslant K_s(n_{c,i}) + c$ for some s, so there are no more than $(c + i + d)2^{c+b}$-many such enumerations. Thus the c strategy eventually stops acting, and for all sufficiently large s, we have $K_s(A_s \restriction n_{c,i}) > K_s(n_{c,i}) + c$, whence $K(A \restriction n_{c,i}) > K(n_{c,i}) + c$. $\square$

It remains to show that A has finite self-information. We begin with an auxiliary lemma.

Lemma 14. *Let i_s^c be the value of the counter associated with c at stage s. There are k and t such that for each c and $s > t$ for which m_{c,i_s^c}, n_{c,i_s^c} are defined, we have*

$$2^{-K_{s+1}^{A_s}(n_{c,i_s^c})} \geqslant 2^{-c-k} \sum_{\sigma \in C(m_{c,i_s^c}, s)} 2^{-K_s^{A_s}(\sigma)}.$$

Proof. We enumerate an oracle KC set L. By Lemma 12, there are k and t such that if $\langle n_i, \sigma_i, \tau_i \rangle$ enters L at stage $s > t$ then $K_{s+1}^{\tau_i}(\sigma_i) \leqslant n_i + k$. At each stage s and for each $c \leqslant s$, proceed as follows. For each σ in $C(m_{c,i_s^c}, s)$, let μ be a witness to σ being in that set; that is, a minimal length A_s-description of σ at stage s with use $> m_{c,i_s^c}$. Put μ into an auxiliary set $S_{c,s}$. Let $u_{c,s}$ be the measure of the open set in Cantor space determined by $S_{c,s}$; in other words, $u_{c,s} = \sum_{\sigma \in C(m_{c,i_s^c}, s)} 2^{-K_s^{A_s}(\sigma)}$. Let j be least such that $\langle j, n_{c,i_s^c}, \tau \rangle$ is currently in L for some $\tau \prec A_s$. If $c - \log u_{c,s} < j$, then put $\langle c - \log u_{c,s}, n_{c,i_s^c}, A_s \rangle$ into L. (Here $\log r$ means the base 2 logarithm of r, rounded up to the nearest largest integer.)

If $s < u$ is such that $A_s \prec A_u$ and $\mu \in S_{c,s}$, then μ is still an A_u-description, since its use as an A_s-description has not changed between stages s and u. Thus, if $\nu \in S_{c,u}$ and $\nu \neq \mu$, then μ and ν are incompatible. This argument shows that for any X, the set $\bigcup_{A_s \prec X} S_{c,s}$ is prefix-free. Thus the sum of $u_{c,s}$ taken over all s such that $A_s \prec X$ is bounded by 1. Hence the sum of 2^{-j} taken over all axioms in L of the form $\langle j, n_{c,i_s^c}, A_s \rangle$ for $A_s \prec X$ is bounded by 2^{-c+1}, and L is an oracle KC set. Thus, by Lemma 12, for each $s > t$ we have

$$K_{s+1}^{A_s}(n_{c,i_s^c}) \leqslant c - \log u_{c,s} + k = c + k - \log \sum_{\sigma \in C(m_{c,i_s^c},s)} 2^{-K_s^{A_s}(\sigma)},$$

which gives us the desired inequality. $\qquad\square$

Lemma 15. *A has finite self-information.*

Proof. We must show that the weight of M is finite, and hence M is actually a KC set. Let t be as in Lemma 14. Let $\widehat{M}$ be the portion of M enumerated after stage t. It is enough to show that the weight of $\widehat{M}$ is finite.

We enumerate $\langle K_s^{A_s}(\sigma) + f_s(\sigma), \sigma \rangle$ into $\widehat{M}$ only when we have $K_s^{A_s}(\sigma) + f_s(\sigma) < K_u^{A_u}(\sigma) + f_u(\sigma)$ for all $u < s$, so the weight of $\widehat{M}$ is bounded by $2 \sum_\sigma 2^{-\min_{u>t}(K_u^{A_t}(\sigma)+f_u(\sigma))}$. For each σ, either

$$\min_{u>t} K_u^{A_u}(\sigma) + f_u(\sigma) = K^A(\sigma) + f(\sigma),$$

or there is an $s > t$ such that

$$K_s^{A_s}(\sigma) + f_s(\sigma) = \min_{u>t} K_u^{A_u}(\sigma) + f_u(\sigma)$$

and there is a minimal length A_s-description of σ whose use is violated by an enumeration into A at stage s. The amount contributed to the weight of $\widehat{M}$ by all σ for which the former case holds is bounded above by $2 \sum_\sigma 2^{-(K^A(\sigma)+f(\sigma))} < 2 \sum_\sigma 2^{-K^A(\sigma)} < 2$, so it is enough to bound the sum of $2^{-(K_s^{A_s}(\sigma)+f_s(\sigma))}$ over all σ and $s > t$ such that some minimal length A_s-description of σ is violated by an enumeration into A at stage s. For each such σ and s, the enumeration into A at stage s is done for the sake of some particular c. Say that c is responsible for $2^{-(K_s^{A_s}(\sigma)+f_s(\sigma))}$. It is enough to fix c and show that the sum of all $2^{-(K_s^{A_s}(\sigma)+f_s(\sigma))}$ for which c is responsible is bounded by $2^{-c+O(1)}$.

Let i_s be the value of the counter associated with c at stage s. As above, let $C(x,s)$ be the set of all σ such that there is a minimal length A_s-description of σ at stage s with use greater than x. The cost

of putting x into A at stage s for the sake of c (in terms of the sum we are trying to bound) is

$$\sum_{\sigma \in C(x,s)} 2^{-K_s^{A_s}(\sigma)-f_s(\sigma)}.$$

Let k be as in Lemma 14. Then, at stages $s > t$ at which the c strategy can act, we have

$$2^{-K_{s+1}(n_{c,i_s})+c} \geqslant 2^{-K_{s+1}^{A_s}(n_{c,i_s})} \geqslant 2^{-c-k} \sum_{\sigma \in C(m_{c,i_s},s)} 2^{-K_s^{A_s}(\sigma)}.$$

So each time we put a number in (m_{c,i_s}, n_{c,i_s}) into A at a stage $s > t$, we add to our sum at most

$$\sum_{\sigma \in C(m_{c,i_s},s)} 2^{-K_s^{A_s}(\sigma)-f_s(\sigma)} \leqslant \sum_{\sigma \in C(m_{c,i_s},s)} 2^{-K_s^{A_s}(\sigma)-4c} \leqslant$$
$$2^{-4c}2^{-K_{s+1}(n_{c,i_s})+2c+k} = 2^{-K_{s+1}(n_{c,i_s})-2c+k}.$$

By Lemma 11, for each i and each value of $K_s(n_{c,i})$, we put at most 2^{c+b}-many numbers in $(m_{c,i}, n_{c,i})$ into A. For each stage s at which we do so, except possibly for the last one, $K_{s+1}(n_{c,i}) = K_s(n_{c,i})$. Thus our total cost is bounded by

$$(2^{c+b} + 1) \sum_i \sum_{j \geqslant K(n_{c,i})} 2^{-j-2c+k} \leqslant 2^{c+b+1} \sum_i 2^{-K(n_{c,i})-2c+k+2} =$$
$$2^{-c+b+k+3} \sum_i 2^{-K(n_{c,i})} < 2^{-c+b+k+3}.$$

Therefore, the weight of $\widehat{M}$ is bounded by $2 + \sum_c 2^{-c+b+k+3}$, which is finite. □

This lemma completes the proof of the theorem. □

3. Relationships to Other Weakness Notions

Recent work has explored the relationship between K-triviality and *jump-traceability*, a notion of approximability of the jump operator. We begin with some definitions.

Definition 16. (i) Define the *jump operator* J by $J^A(e) = \Phi_e^A(e)$, where Φ_e is the eth partial computable function.
(ii) A is *jump-traceable* if there are a computable function h and a uniformly c.e. family $\{T_e\}$ of subsets of ω such that $\forall e \, |T_e| \leqslant h(e)$ and $J^A(e)\!\downarrow \, \Rightarrow \, J^A(e) \in T_e$. If h is known we say A is jump-traceable *via* h.

(iii) A function h is an *order* if it is computable, non-decreasing, and unbounded.

(iv) A is *strongly jump-traceable* if it is jump-traceable via every order.

Figueira, Nies, and Stephan [7] constructed a promptly simple strongly jump-traceable set, separating that class from the computable sets. Cholak, Downey, and Greenberg [4] showed that the class of strongly jump-traceable c.e. sets is properly contained in the class of K-trivial c.e. sets. Recently, Diamondstone, Greenberg, and Turetsky [5] showed that every strongly jump-traceable set is computable in a c.e. strongly jump-traceable set, which implies that the result in [4] holds without the assumption of computable enumerability.

The proof in [4] shows that if a c.e. set A is jump-traceable via an order $h \approx \sqrt{\log n}$ then it is K-trivial. Nies [18] showed that all K-trivials are jump-traceable, and in particular that there exists a constant C such that if A is low for K with constant b, then A is jump-traceable with trace obtained effectively in b and with bound $C2^b n \log n$. This result led to the question of whether between $\sqrt{\log n}$ and $n \log n$ there is a growth rate cutoff that separates K-trivials from non-K-trivials. Hölzl, Kräling, and Merkle [9] did find such a growth rate, but not a computable one. Their result implies that every K-trivial is jump-traceable via an order that is $O(\log n)$. Recently, Turetsky [22] built a K-trivial that is not jump-traceable via any order that is $o(\log n)$.

Cholak, Downey, and Greenberg [4] asked in particular whether the K-trivials are exactly those sets that are h-jump-traceable for all *convergent* order functions h, those functions for which $\sum_n \frac{1}{h(n)} < \infty$. Containment of the K-trivials in this class of jump-traceables is a consequence of the decanter method used to show that all K-trivials are superlow. (See [6] or [20] for details on this method.) Barmpalias, Downey, and Greenberg [2], however, showed that this containment is proper. Hence K-triviality sits strictly between "jump-traceable for all order functions h" (i.e., strongly jump-traceable) and "jump-traceable for all convergent order functions h" (in fact "jump-traceable for all superlinear order functions", where *superlinear* means $\lim_n \frac{h(n)}{n} = \infty$).

We may ask where the sets that have finite self-information fall in this hierarchy. The following result and its proof are a first step in addressing this question.

Proposition 17. *If a set has finite self-information then it is jump-traceable.*

Proof. Suppose A is not jump-traceable. Let $\sigma_0, \sigma_1, \dots$ be an effective listing of $2^{<\omega}$. If $J^A(e)\!\downarrow$ then let $\tau_e = \sigma_{J^A(e)}$. Let L be the set of all pairs

$\langle K_s(e), \tau_e \rangle$ such that $s \in \omega$ and τ_e is defined. It is easy to check that L is a KC set relative to A, so if τ_e is defined then $K^A(\tau_e) \leqslant^+ K(e)$.

Let $T_e = \{i \mid K(\sigma_i) < 4 \log e\}$. The T_e are uniformly c.e. and $|T_e| < e^4$. Since A is not jump-traceable, there must be infinitely many e such that $J^A(e) \notin T_e$. For any such e, we have $K(\tau_e) \geqslant 4 \log e$. Let S be the set of such e. Since $K(e) \leqslant 2 \log e + O(1)$, we have

$$I(A : A) > \log \sum_{e \in S} 2^{K(\tau_e) - 2K^A(\tau_e) - O(1)} \geqslant \log \sum_{e \in S} 2^{4 \log e - 2K(e) - O(1)} = \infty.$$

$\square$

From the proof we see that if a set has finite self-information then it is jump-traceable via the order n^4. Since $K(e) \leqslant \log e + 2 \log \log e + O(1)$, we may replace the $4 \log e$ in the definition of T_e by $2 \log e + 4 \log \log e$, so in fact, if a set has finite self-information then it is jump-traceable via any $n^{2+\epsilon}$ for $\epsilon > 0$. We do not know whether every set that has finite self-information is jump-traceable via n^2, say, so our results leave open the exact relationships between having finite self-information and various interesting subclasses of the jump-traceables, such as the class of sets that are jump-traceable for all convergent order functions.

Recall that a set A is *generalized low*, or GL_1, if $A' \leqslant_{\mathrm{T}} A \oplus \emptyset'$. It follows from the above result that if a set A has finite self-information, then it is GL_1 (see Proposition 8.4.3 of [20]). If A is also Δ_2^0, it is therefore low, in the usual computability theoretic sense. However, Herbert [8] has recently shown that there is a perfect Π_1^0 class of sets that have finite self-information, which in particular answers a question in a previous version of this paper by showing the existence of non-Δ_2^0 sets with finite self-information.

Lutz [16] and others have developed an effective version of Hausdorff dimension based on martingales in inflationary environments, called *s*-gales. From success of *s*-gales we have the *constructive dimension*, or *effective Hausdorff dimension*, $\dim(S)$, and from strong success of *s*-gales we have *constructive strong dimension*, or *effective packing dimension*, $\mathrm{Dim}(S)$. (See Chapter 13 of [6] for more details.) These dimensions have a deep tie to randomness, as shown by the following results.

Theorem 18. *Let $S \in 2^\omega$.*

(i) (Mayordomo [17]). $\dim(S) = \liminf\limits_{n \to \infty} \dfrac{K(S \restriction n)}{n}$.

(ii) (Athreya, Hitchcock, Lutz, and Mayordomo [1]).
$$\mathrm{Dim}(S) = \limsup_{n \to \infty} \frac{K(S \upharpoonright n)}{n}.$$

Effective dimension and the theorem above may be relativized, giving rise to the notion of lowness for dimension. A set A is *low for effective Hausdorff dimension* if the effective Hausdorff dimension of S relative to A is equal to the effective Hausdorff dimension of S for all S. We can define lowness for effective packing dimension similarly. Clearly, all K-trivial sequences are low for both notions of dimension. In the proof of Theorem 6, if instead of ensuring $K(\sigma) \leqslant^+ K^A(\sigma) + f(\sigma)$ for the f of Lemma 7 we instead ensure it for $f(\sigma) = \log |\sigma|$, then for all S,
$$\frac{K(S \upharpoonright n)}{n} \leqslant^+ \frac{K^A(S \upharpoonright n)}{n} + \frac{\log n}{n},$$
and the last fraction has $\limsup$ and $\liminf$ zero. Hence, a set A built in this way will be low for both effective Hausdorff and effective packing dimension, but not K-trivial, yielding the following result.

Theorem 19. *There is a c.e. set that is low for both effective Hausdorff dimension and effective packing dimension but is not K-trivial.*

In the forthcoming paper [11], Lempp, Miller, Ng, Turetsky, and Weber show that there is in fact a perfect Π_1^0 class of sets that are low for effective Hausdorff dimension, a result that also follows by the methods of Herbert [8].

The function f from Lemma 7 grows much more slowly than $\log |\sigma|$, so it seems to be much easier to produce a set that is low for dimension than one that has finite self-information. On the other hand, Lempp, Miller, Ng, Turetsky, and Weber [11] show that if a set is low for effective Hausdorff dimension then it is jump traceable via $O(h^r)$ for any convergent order function h and any $r > 1$, and thus, in particular, via $n^{1+\epsilon}$ for any $\epsilon > 0$, which is a better bound than what we obtained above for sets that have finite self-information.

Question 20. What is the exact relationship between having finite self-information and lowness for dimension?

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