Immunity for Closed Sets*

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Abstract. The notion of immune sets is extended to closed sets and Π^0_1 classes in particular. We construct a Π^0_1 class with no computable member which is not immune. We show that for any computably inseparable sets A and B, the class $S(A, B)$ of separating sets for A and B is immune. We show that every perfect thin Π^0_1 class is immune. We define the stronger notion of *prompt immunity* and construct an example of a Π^0_1 class of positive measure which is promptly immune. We show that the immune degrees in the Medvedev lattice of closed sets forms a filter. We show that for any Π^0_1 class P with no computable element, there is a Π^0_1 class Q which is not immune and has no computable element, and which is Medvedev reducible to P. We show that any random closed set is immune.

Keywords: Computability, Π^0_1 Classes

1 Introduction

The notion of a simple c.e. set and the corresponding complementary notion of an immune co-c.e. set are fundamental to the study of c.e. sets and degrees. Together with variations and related notions such as *effectively immune*, *promptly simple*, *hyperimmune* and so forth, they permeate the classic text of R.I. Soare [21] and its updated version.

Many of the results on c.e. sets and degrees have found counterparts in the study of effectively closed sets (Π^0_1 classes). See the surveys [10, 11] for examples. In particular, hyperhyperimmune co-c.e. sets correspond to thin Π^0_1 classes [7–9] and hyperimmune co-c.e. sets correspond to several different notions including *smallness* studied by Binns [5, 6].

In this paper we consider the notion of immune sets as applied to Π^0_1 classes and closed sets in general. We work in $2^{\mathbb{N}}$ with the topology generated by basic clopen sets called *intervals*. For any $\sigma \in \{0, 1\}^*$ the interval $I(\sigma)$ is $\{X : \sigma \subset X\}$, where here $\subset$ means initial segment. Notation is standard; we note that $\sigma \upharpoonright n$ is the length-n initial segment of σ, and if $T \subseteq \{0, 1\}^*$ is a tree (i.e., it is closed

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under initial segment), $[T] \subseteq 2^{\mathbb{N}}$ denotes the set of infinite paths through T. For any set $P \subseteq 2^{\mathbb{N}}$, we may define the tree $T_P = \{\sigma \in \{0,1\}^* : I(\sigma) \cap P \neq \emptyset\}$; the closed sets $P \subseteq 2^{\mathbb{N}}$ are exactly those for which $P = [T_P]$. A Π^0_1 class is a closed set for which some computable tree $T \supseteq T_P$ has $[T] = P$; in this case T_P is a Π^0_1 set. For any tree T, let $Ext(T)$ be the set of nodes of T which have an infinite extension in $[T]$, thus if $P = [T]$, then $Ext(T) = T_P$.

An infinite set $C \subseteq \omega$ is said to be *immune* if it does not include any infinite c.e. subset, or equivalently if it has no infinite computable subset. A c. e. set which is the complement of an immune set is said to be *simple*. We say that a closed set $P \subseteq 2^{\mathbb{N}}$ is immune if T_P is immune.

It is easy to see that an immune closed set has no computable member. We will construct in section 2 a Π^0_1 class which is *not* immune and still has no computable member. We will show that for any computably inseparable sets A and B, the class $S(A, B)$ of separating sets for A and B is immune. We will show that every perfect thin Π^0_1 class is immune. We define the stronger notion of *prompt immunity* and construct an example of a Π^0_1 class of positive measure which is promptly immune.

In section 3, we consider connections between immunity and Binns' notion of *smallness* [5] and also connections with the Medvedev degrees of difficulty [16, 19]. We show that for closed sets P and Q, the meet $P \oplus Q$ is immune if and only if both P and Q are immune, whereas the join $P \otimes Q$ is immune if and only if at least one of P and Q are immune. We show that for any Π^0_1 class P with no computable element, there is a non-immune Π^0_1 class Q with no computable element which is Medvedev reducible to P.

In section 4, we show that any random closed set (in the sense of [2]) is immune. We also show that any random closed set is not small.

2 Immunity for Π^0_1 classes

We begin with some basic results. The following is a useful additional characterization of immunity.

Lemma 1. *P is not immune if and only if there is a computable sequence $\{\sigma_n : n \in \omega\}$ such that $\sigma_n \in T_P \cap \{0,1\}^n$ for each n.*

Proof. The reverse implication is immediate. Now suppose that C is an infinite computable subset of T_P and enumerate C as $\{\tau_0, \tau_1, \dots\}$. Observe that C must have arbitrarily long elements and define σ_n to be $\tau_i \upharpoonright n$, where i is the least such that $|\tau_i| \geq n$.

We often refer to a Π^0_1 class with no computable members as a *special* Π^0_1 class. The following two results shows that the immune classes are a proper subset of the special classes.

Proposition 1. *If P is immune, then P is special.*

Proof. If P has a computable member X, then $\{X \upharpoonright n : n \in \mathbb{N}\}$ is an infinite computable subset of T_P.

Theorem 1. *There exists a special Π^0_1 class P that is not immune.*

Proof. We will build a sequence of nested computable trees T_s such that $T_P = \cap_s T_s$ and a prefix-free set A such that $A_s = \{\sigma_0, \dots, \sigma_s\} \subseteq Ext(T_s)$ and $|\sigma_s| \geq s$. We have the following requirements:

$$N_e : \varphi_e \text{ total } \Rightarrow W_e \notin P.$$

Each N_e has an associated $m_s(e)$, the minimum length of convergence of φ_e required before we act for N_e. For all e, $m_0(e) = 2e + 1$.

To meet a single requirement N_0 we act as follows. We wait until $\varphi_0(0)\downarrow$, and if that happens at stage s we let $m_s(0) = 1 + \max\{|\sigma_i| : i < s\}$ and choose all σ_t, $t \geq s$, to be incompatible with $\varphi_0(0)$. Then if at stage $s' > s$ we see $\varphi_0 \upharpoonright m_s(0)\downarrow$, we let $T_{s'+1} = T_{s'} - \{\varphi_0 \upharpoonright m_s(0)^\frown\tau : \tau \in \{0,1\}^*\}$. The same module holds for all other requirements; we enumerate a set R of indices of requirements that must be avoided by A. Each $m_s(e)$ changes its value at most once, and the second value it takes on is sufficiently large that the standard measure argument shows $P \cap [\sigma_i] \neq \emptyset$ for each i.

Stage 0: $\forall e\; m_0(e) = 2e + 1$; $A_0 = R_0 = \emptyset$; $T_0 = \{0,1\}^*$.

Stage $s > 0$: For each $e \leq s$ such that $\varphi_e \upharpoonright (2e+1)\downarrow$ newly at s, set $m_s(e) = 2e + 1 + \max\{|\sigma_i| : i < s\}$. Enumerate all such e into R_s. For the rest, let $m_s(e) = m_{s-1}(e)$.

Next, if any $e \leq s$ is such that $\varphi_e \upharpoonright m_s(e) \downarrow = \tau_e \in T_{s-1}$, let $T_s = T_{s-1} - \{\tau_e\rho : \rho \in \{0,1\}^*, e \text{ as above}\}$. Otherwise let $T_s = T_{s-1}$.

Finally, let Q be the part of T_s uncovered by A and R. That is,

$$Q = T_s - \{\tau^\frown\rho : \tau \in A_{s-1} \cup \{\varphi_e \upharpoonright (2e+1) : e \in R\}, \rho \in \{0,1\}^*\}.$$

Note that we would get the same Q if we replaced T_s by $\{0,1\}^*$. Choose the leftmost $\sigma \in Q$ of length at least $s + 2$ and let it be $\sigma_s \in A_s$.

To verify the construction works, first note every σ_i has an extension by a straightforward measure argument: if $\varphi_e \upharpoonright (2e+1)\downarrow$ at or before stage i, σ_i will be chosen to avoid it; if $\varphi_e \upharpoonright (2e+1)\downarrow$ after stage i it will be allowed to remove at most $2^{-2e-1-|\sigma_i|}$ from the tree. The sum of the measure so eliminated is bounded by $\frac{2}{3}\mu([\sigma_i])$. Second, another measure argument shows there is always enough room in Q to choose a new string in A without covering all of T_s. The measure of Q at stage s is at least

$$x = 1 - \sum_{e=0}^{s} 2^{-2e-1} - \sum_{i=1}^{s-1} 2^{-i-2},$$

which we need to be greater than (at most) 2^{-s-2}. It is easily checked that $x - 2^{-s-2}$ is

$$\frac{1}{12} + \frac{1}{3 \cdot 2^{2s+1}} + \frac{1}{2^{s+2}},$$

which is clearly positive.

Since it is clear that the requirements are met, P is a Π^0_1 class, and $A \subset T_P$ is computable, the proof is complete.

The next results show many Π^0_1 classes of interest are immune. Recall $S(A, B)$ denotes the class of separating sets for A and B (all C such that $A \subseteq C$ and $B \cap C = \emptyset$); it is a closed set, and when A and B are c.e. it is a Π^0_1 class.

Proposition 2. *If A and B are computably inseparable, then $S(A, B)$ is immune.*

Proof. Suppose that $W \subset T_{S(A,B)}$ is an infinite c.e. set, enumerated without repetition as $\sigma_0, \sigma_1, \ldots$. Note that for any $\sigma \in W$ and any $n < |\sigma|$, $n \in A \Rightarrow \sigma(n) = 1$ and $n \in B \Rightarrow \sigma(n) = 0$. Since W must have elements of arbitrary length, we may computably define $i(n)$ to be the least i such that $|\sigma_i| > n$ and let $X(n) = \sigma_{i(n)}(n)$ to compute a separating set for A and B.

The notion of a *thin* Π^0_1 class corresponds to that of a hyperhyperimmune set and has been studied extensively by many researchers in articles including [7–9].

Since any hyperhyperimmune set is also immune, the following result is expected.

Proposition 3. *If P is a perfect thin Π^0_1 class, then P is immune.*

Proof. Let P be perfect thin (and therefore having no computable member) and suppose that some computable set $W \subseteq T_P$. Then $T_P - W$ is a Π^0_1 set, so that $[T_P - W]$ is a Π^0_1 subclass of P and hence there exists a clopen set U such that $[T_P - W] = P \cap U$. It follows that $T_{P \cap U} \subseteq T_P - W$. We claim that without loss of generality $T_P - W = T_{P \cap U}$. That is, let $Q = P - U$ and consider $T_Q - W$. $T_Q - W \subseteq T_P - W$, so that $[T_Q - W] \subseteq P \cap U$. On the other hand, $T_Q - W \subseteq T_Q$, so $[T_Q - W] \subseteq Q = P - U$. Thus $[T_Q - W]$ is empty and hence $T_Q - W$ is finite. Thus we may assume without loss of generality that $T_Q \subseteq W$ and therefore $T_P - W \subset T_P - T_Q = T_{P \cap U}$. It follows that $T_P - W = T_{P \cap U}$. This means that in fact $W = T_{P-U}$ which means that T_{P-U} is computable. But P has no computable members, so that $P - U = \emptyset$ and therefore W is finite.

For any c.e. set A and $n \in \omega$, let A_n denote as usual the elements which have been enumerated into A by stage n; A is said to be *promptly simple* if there is a computable function π such that for any infinite c.e. set $W_e \subseteq \omega$, there exist n, s such that $n \in W_{e,s+1} - W_{e,s}$ and $n \in A_{\pi(s)}$.

For P a Π^0_1 class, let T be a computable tree giving P. For each s, let T_s be the collection of nodes of T which have length-s extensions in T. Let $\{\sigma_n\}_{n\in\mathbb{N}} = \{\langle\rangle, 0, 1, 00, 01, 10\ldots\}$ denote the length-lexicographical ordering of the elements of $\{0,1\}^*$. We say that P is *promptly immune* if there is a computable function π such that for any infinite c.e. set W, there exist n, s such that

$$n \in W_{s+1} - W_s \ \ \& \ \ \sigma_n \notin T_{\pi(s)}.$$

There exist Π^0_1 classes with positive measure which have no computable elements. The next result is an improvement on this.

Proposition 4. *There exists a Π^0_1 class P of positive measure which is promptly immune.*

Proof. We define the Π^0_1 class $P = [T]$ in stages T_s and let $T = \bigcap_s T_s$. P will be promptly immune via the function $\pi(s) = s+1$. For each e, we will wait for some n such that $|\sigma_n| > 2e$ to come into W_e at stage $s+1$ and then remove σ_n from T_{s+1} by removing σ_n and all extensions (if any) from T. Initially $T_0 = \{0,1\}^*$. After stage s, we will have satisfied some of the requirements. At stage $s+1$, we look for the least $e \leq s$ which has not yet been satisfied and such that some suitable $n \in W_{e,s+1} - W_{e,s}$. We meet this requirement by setting $T_{s+1} = T_s - \{\tau : \sigma_n \subseteq \tau\}$. Not that this action removes from $[T]$ a set of measure $\leq 2^{-2e-1}$, so that the total measure removed is

$$\leq \sum_e 2^{-2e-1} \leq \frac{2}{3}.$$

It follows that $T_s \neq \emptyset$ for any s and therefore $P = [T]$ is not empty, and in fact has measure at least $\frac{1}{3}$.

3 Immunity and smallness

Π^0_1 classes are often viewed as collections of solutions to some mathematical problem. Muchnik and Medvedev reducibility, defined for closed subsets of $2^{\mathbb{N}}$ and indeed $\mathbb{N}^{\mathbb{N}}$ in general, order classes based on this viewpoint. The class A is *Muchnik* (a.k.a. *weakly*) *reducible* to the class B ($A \leq_w B$) if for every $X \in B$ there is $Y \in A$ such that $Y \leq_T X$ [17]. The class A is *Medvedev* (a.k.a. *strongly*) *reducible* to B if there is a single Turing reduction procedure which, when given any element of B as an oracle, computes an element of A; it is exactly the uniformization of Muchnik reduction [16]. These reductions have been studied extensively by Binns (e.g., [4]) and Simpson (e.g., [20]), among others, and have connections to randomness [18].

For $X, Y \in 2^{\mathbb{N}}$, the join $X \oplus Y = Z$, where $Z(2n) = X(n)$ and $Z(2n+1) = Y(n)$. Similarly for finite sequences σ and τ of equal length, we may define $\sigma \oplus \tau = \rho$, where $\rho(2n) = \sigma(n)$ and $\rho(2n+1) = \tau(n)$.

The quotient structure of the Π_1^0 classes under either Muchnik or Medvedev equivalence is a lattice, and both have the same join and meet operators. The join of P and Q is given by

$$P \otimes Q = \{X \oplus Y : X \in P, Y \in Q\}.$$

If $P = [S]$ and $Q = [T]$, then $P \otimes Q = S \otimes T$, where

$$S \otimes T = \{\sigma \oplus \tau, (\sigma \oplus \tau)i : \sigma \in S, \tau \in T, i \in \{0, 1\}\}.$$

The meet is given by

$$P \oplus Q = \{0^\frown X : X \in P\} \cup \{1^\frown Y : Y \in Q\}.$$

Theorem 2. *For any closed sets P and Q, $P \oplus Q$ is immune if and only if both P and Q are immune.*

Proof. Suppose first that P is not immune and let $C \subseteq T_P$ be an infinite computable set. Then $\{0^\frown\sigma : \sigma \in C\}$ is a computable subset of $T_{P\oplus Q}$. A similar argument holds if Q is not immune.

Next suppose that $P \oplus Q$ is not immune and let $C \subseteq T_{P\oplus Q}$ be an infinite computable set. Let $C_i = \{\sigma : i^\frown\sigma \in C\}$ for $i = 0, 1$. Then $C_0 \subseteq T_P$, $C_1 \subseteq T_Q$ and both sets are computable. Clearly either C_0 is infinite or C_1 is infinite, which implies that either P is not immune or Q is not immune.

Theorem 3. *For any closed sets P and Q, $P \otimes Q$ is immune if and only if at least one of P and Q are immune.*

Proof. Suppose first that $P \otimes Q$ is not immune and by Lemma 1 let $C = \{\rho_0, \rho_1, \dots\}$ be a computable subset of $T_{P\otimes Q}$ with $|\rho_n| = n$ for each n. Then for each n, $\rho_{2n} = \sigma_n \oplus \tau_n$ with $\sigma_n \in T_P$ and $\tau_n \in T_Q$, showing both P and Q are not immune.

Next suppose that both P and Q are not immune and let $C_0 = \{\sigma_0, \sigma_1, \dots\} \subset T_P$ and $C_1 = \{\tau_0, \tau_1, \dots\} \subset T_Q$ with $|\sigma_n| = |\tau_n| = n$ for each n. Then $\{\sigma_n \oplus \tau_n : n \in \mathbb{N}\}$ is an infinite computable subset of $T_{P\otimes Q}$, so $P \otimes Q$ is not immune.

We may compare immunity with other "smallness" notions for Π_1^0 classes. Binns [5] defined a *small* closed set P to be one such that there is no computable function g such that, for all n, $card(\{0, 1\}^{g(n)} \cap T_P) > n$. He showed that $P \oplus Q$ and $P \otimes Q$ are each small if and only if both P and Q are small, which immediately distinguishes immunity and smallness.

Another distinction occurs in the maximum degree of each lattice. Recall the family $S(A, B)$ of separating sets of c.e. sets A, B is a Π_1^0 class. An important example is the class DNC_2, the set of diagonally noncomputable functions; here

$A = \{e : \varphi_e(e) = 0\}$ and $B = \{e : \varphi_e(e) = 1\}$. DNC_2 has maximum Medvedev and Muchnik degree, and by Proposition 2 it is immune. However, Binns has proved that DNC_2 is not small. In fact, all Medvedev complete Π^0_1 classes are immune and not small, as they are all computably homeomorphic.

We look next for a containment relationship between smallness and immunity. Smallness alone will not give immunity because a Π^0_1 singleton (i.e., a computable path) is small. Binns' original special small class [5] is $S(A, B)$ for computably inseparable A, B, such that $A \cup B$ is hypersimple; by Proposition 2 it is immune. Our construction above in Theorem 1 of a special nonimmune class produces a class of positive measure, which is therefore not small. We have the following question:

Question 4. *If P is small and special, is P necessarily immune?*

We now turn to questions of density. Let 0_M denote the least Medvedev degree, which consists of all Π^0_1 classes that have a computable member. Binns has shown there is a nonsmall class of every nonzero Medvedev degree. We have the following bounding result for nonimmune classes.

Theorem 5. *For any nonzero Π^0_1 class P, there is a Π^0_1 class Q with $0_M <_M Q \leq_M P$ which is not immune.*

Proof. Let R be the Π^0_1 class of Theorem 1 which is nonzero and also not immune. It follows from Theorem 2 that $P \oplus R$ is not immune, but it is also special and certainly $P \oplus R \leq_M P$.

Question 6. *Does every nonzero Medvedev degree contain an immune Π^0_1 class?*

Ambos-Spies et al [1] showed that any promptly simple c.e. degree cups to $\mathbf{0}'$; in fact they had the much stronger result that the promptly simple degrees are exactly the c.e. degrees that are cuppable by a low c.e. degree. We have the following more modest Cupping Conjecture.

Conjecture 7. *If P is promptly immune, then there exists Q, not Medvedev complete, such that $P \otimes Q$ is Medvedev complete.*

4 Immunity and randomness

Finally we consider the immunity of random closed sets. A closed set P may be coded as an element of $3^{\mathbb{N}}$; P is called random if that sequence is Martin-Löf random (for background on randomness see [12]). The code of P is defined from T_P; the nodes of T_P are considered in order by length and then lexicographically, and each one is represented in the code by 0, 1, or 2 according to whether the node has only the left child, only the right child, or both children, respectively.

Randomness for closed sets is defined and explored in [2, 3], where it is shown among other results that no Π^0_1 class is random, and that no random closed set contains an f-c.e. path for any computable f bounded by a polynomial. The following theorem does not follow immediately but is not surprising.

Theorem 8. *If P is a random closed set, then P is immune.*

Proof. Fix a computable sequence $C = (\sigma_1, \sigma_2, \dots)$ such that $|\sigma_n| = n$ for each n. For $n > 0$, let $S_n = \{Q : (\forall i \leq n)\ \sigma_i \in T_Q\}$. Then S_n is a clopen set in the space of closed sets and the sequence $\{S_n : n \in \omega\}$ is uniformly c.e. It is clear that $C \subseteq T_P$ if and only if $P \in S_n$ for all n. Now consider the Lebesgue measure $\lambda(S_n)$. Certainly $\lambda(S_1) = 2/3$. Given $\lambda_n = \lambda(S_n)$ and σ_{n+1}, let $i \leq n$ be the largest such that $\sigma_i \subset \sigma_{n+1}$. Then $\lambda_{n+1} = (\frac{2}{3})^{n+1-i}\lambda_n \leq \frac{2}{3}\lambda_n$. Hence $\lambda(S_n) \leq (\frac{2}{3})^n$ for each n. It follows that $\{S_{2n} : n \in \omega\}$ is a Martin-Löf test and hence no random closed set can belong to every S_n. Hence if P is random, then C is not a subset of T_P. Since this holds for every such C, it follows that random closed sets are immune.

Since a random ternary sequence must contain $\frac{1}{3}$ 2s in the limit, intuitively the tree it codes must branch too much to be small. This is a straightforward consequence of the following, which is drawn from Lemma 4.5 in [2].

Lemma 2. *Let Q be a random closed set. Then there exist a constant $C \in \mathbb{N}$ and $k \in \mathbb{N}$ such that for all $m > k$,*

$$C\left(\frac{4}{3}\right)^m\left(1 - m^{-\frac{1}{4}}\right) \;<\; card(T_Q \cap \{0,1\}^m) \;<\; C\left(\frac{4}{3}\right)^m\left(1 + m^{-\frac{1}{4}}\right).$$

Corollary 9. *If Q is a random closed set, Q is not small.*

Proof. For C, k as in Lemma 2, define the function $g(n)$ as

$$g(n) = \max\left\{k+1, \min\left\{m : n < C\left(\frac{4}{3}\right)^m\left(1 - m^{-\frac{1}{4}}\right)\right\}\right\}.$$

It is clear that g is computable, and by Lemma 2, for all n the number of branches at level $g(n)$ will be at least n.

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