Tutorial on Π^0_1 Classes

Rebecca Weber, Dartmouth/Florida

Plan:

- Basic definitions; examples
- Basis and antibasis theorems
- Connections to randomness
- Enumeration and index sets
- Lattice intervals and invariance
- Lattice embeddings and theories

Definitions

ω: natural numbers, beginning at 0 (k, n, m)

$2^{<\omega}$: set of all finite binary sequences (a.k.a. $\{0,1\}^*$); complete binary-branching tree (σ, τ, ρ)

2^{ω}: set of all infinite binary sequences; Cantor space (X, Y, Z)

subtree of $2^{<\omega}$: subset closed under initial segment; dead ends allowed

Π^0_1 *class*: set of infinite paths through a computable subtree of $2^{<\omega}$

Definitions

Empty string denoted λ (often also written $\langle\rangle$)

Length of σ is $|\sigma|$

$0^n, 1^n, 0^\omega, 1^\omega$: string of all 0s or all 1s of length in superscript

Concatenation of σ and τ indicated by $\sigma\tau$ or $\sigma^\frown\tau$.

If τ extends σ $(\exists\rho(\sigma\rho = \tau))$, write $\sigma \subseteq \tau$. If $\sigma \not\subseteq \tau$ and $\tau \not\subseteq \sigma$, $\sigma \perp \tau$.

Definitions

$X \upharpoonright i$ is the length-i initial segment of X

If $|\sigma| = n$, $\sigma(0)$ is the first bit of σ and $\sigma(n-1)$ the last.

$[\cdot]$ means "infinite strings associated with" for us: If T is a tree, $[T]$ is the associated Π^0_1 class; if σ is a finite string, $[\sigma]$ is the set of all infinite strings that extend σ (*interval* around σ)

Lattice of all Π^0_1 classes: denoted $\mathcal{E}_\Pi$.

Working within 2^ω

Topology: basic clopen sets are intervals.

Measure: size of interval $[\sigma]$ is $2^{-|\sigma|}$ (coin-toss probability measure).

Metric: distance between X and Y is measure of least interval containing both; i.e., $2^{-|\sigma|}$ for σ the longest initial segment common to X and Y.

Solution sets

Perhaps the most significant use of Π^0_1 classes is as representations of solution sets to problems of finding examples of something (e.g. separating sets, ideals, zeros of a function). When the problem is presented as a computably enumerable sequence of computable requirements we can often build a Π^0_1 class with paths corresponding exactly to solutions of the problem.

Note these proofs need not be effective – there may be no solution that is computable. If we collect the solutions into a Π^0_1 class, we may be able to make other complexity-related statements about them, though.

Template for building a Π^0_1 class

– Start with λ, the empty node.

– Unless otherwise instructed, at stage $s+1$ enumerate both children of every length-s node in the tree.

– Concurrently enumerate a list of properties the infinite paths must have.

– Cease extending any node when you see all sequences in its interval will fail some property.

– To survive at all levels, a path must satisfy all properties.

– Why computable? Only put nodes in, never take them out, and all length-s nodes are in at stage s.

Example 1: Separating sets (the canonical example)

Have disjoint c.e. A and B. Associate elements of ω with levels of the tree, starting at level 1; paths are interpreted as characteristic functions.

Requirement on paths X: if $n \in A$, $X(n) = 1$. If $n \in B$, $X(n) = 0$. Enumeration of A and B gives enumeration of desired properties.

Pruning method: If n enters A at stage s, cease extending any living length-s node σ such that $\sigma(n) = 0$. Likewise for B and $\sigma(n) = 1$.

Correct Π^0_1 class: Since any number entering A or B must enter at some finite stage, at that stage all paths containing the wrong level-n value will be killed.

Example 2: Zeros of a computable function

Have computable $f : 2^\omega \to 2^\omega$, presented by enumeration of pairs of intervals $\langle[\sigma_n], J_n\rangle$ ($\{\sigma_n\}$ enumeration of all finite strings in lexicographical order) such that $f\,[[\sigma_n]] \subseteq J_n$ and if $\{X\} = \lim_i \sigma_i$, $\{f(X)\} = \bigcap_i J_i$. Paths of the tree interpreted as elements of 2^ω.

Requirement on paths X: $f(X) = 0^\omega$.

Pruning method: If $\langle[\sigma_s], J_s\rangle$ is such that $0^\omega \notin J_s$, cease extending any living length-s node extending σ_s.

Correct Π^0_1 class: If $f(X) \neq 0^\omega$, then $f(X) = Y$ for some Y a nonzero distance from 0^ω. At some finite stage the sequence of intervals intersecting to $\{Y\}$ will be small enough to exclude 0^ω.

Likewise:

- Fixed points of a computable function (prune when the preimage and image intervals are disjoint)
- Points at which the computable function attains a maximum [minimum] (when you see $\langle I_1, J_1 \rangle, \langle I_2, J_2 \rangle$ such that all elements of J_1 are strictly less [greater] than all elements of J_2, prune I_1)

Other examples

- Complete consistent extensions of an axiomatizable first-order theory (i.e., one whose true sentences form a c.e. set): levels of tree correspond to all sentences in language; prune when you see inconsistency.
- Prime ideals of a c.e. commutative ring with unity: levels of tree correspond to all elements of ring; prune when you see 1s at levels a, b and 0 at level $a + b$, or 1 at level a and 0 level ab for some b, or 0s at levels a, b and 1 at level ab (need commutativity for that characterization of primality).

Completions of PA

PA, or Peano Arithmetic, is a first-order formalization of arithmetic consisting of $=$, $+$, $\cdot$, 0, successor, and induction.

PA is axiomatizable so its completions form a Π^0_1 class.

Solovay & Scott proved the degrees of consistent extensions of PA and completions of PA coincide with each other and with the degrees $\boldsymbol{a}$ such that every Π^0_1 class contains a path of degree $\leq_T \boldsymbol{a}$ (these are the PA degrees, denoted $\boldsymbol{a} >> \mathbf{0}$).

DNR_2

Let $\{\varphi_e\}_{e\in\omega}$ be an enumeration of partial computable functions. $X \in k^\omega$ is *diagonally non-recursive* (DNR_k) if $(\forall n)[X(n) \neq \varphi_n(n)]$.

The DNR_2 sets form a Π^0_1 class (whenever you see convergence of a new computation of $\varphi_n(n)$, prune paths that agree with it at level n).

The Turing degrees of paths of DNR_2 are the same as those of PA, but DNR_2 is a separating class.

Steve will tell more about DNR_2 in the context of Medvedev and Muchnik degrees.

We should note that in some cases *every* Π^0_1 class represents a solution set for some instantiation of a given problem, and in some cases not.

For example:

Not every Π^0_1 class is a separating class, clearly: need only two length-n nodes with different branching properties.

Representability theorems

Every Π^0_1 class represents

- the set of zeros of some computable function (can build the function out of the tree).
- the set of fixed points of some computable function.
- the set of points at which some computable function attains its minimum [maximum].
- the set of complete consistent extensions of some axiomatizable theory.
- the set of prime ideals of some c.e. commutative ring with unity.

A few basis theorems (Jockusch and Soare, 1972)

Every nonempty Π^0_1 class $P \subseteq 2^\omega$ contains

(a) a path of low Turing degree;

(b) a path of c.e. Turing degree;

(c) a computable path or two paths with degree infimum zero;

(d) a path of hyperimmune-free degree.

Consequences of basis theorems

A computable function need not have a computable zero, but it must have a zero of low degree and one of c.e. degree. If it has no computable zeros it has two zeros which form a minimal pair in the Turing degrees.

Likewise a pair of c.e. sets must have a c.e. separating set (this is clear anyway) and a low separating set.

One proof: Low basis theorem (forcing with Π^0_1 classes)

Given $P = [T]$ for computable T, define a sequence of computable subtrees $T = T_0 \supseteq T_1 \supseteq \ldots$ so $\bigcap_e [T_e]$ is nonempty and contains only low paths.

By induction, assume T_e is defined and infinite. Let $U_e = \{\sigma : \Phi^\sigma_{e,|\sigma|}(e)\uparrow\}$ (standard enumeration of functionals Φ_e); U_e is a computable tree. Using $\mathbf{0'}$, choose $T_{e+1} = T_e$ if $U_e \cap T_e$ is finite, and $T_{e+1} = U_e \cap T_e$ otherwise. Hence in T_{e+1} either all paths X give $\Phi^X_e(e)\uparrow$ or all give $\Phi^X_e(e)\downarrow$, and all T_e are infinite so $\bigcap_e [T_e] \neq \emptyset$ by compactness. The construction is computable in $0'$, so $X' \leq_T 0'$ for all $X \in \bigcap_e [T_e] \subseteq [T]$.

Antibasis theorems

- A nonempty Π^0_1 class need not have a computable member (Kreisel 1953)
- The Π^0_1 class with no computable member may even have positive measure, though its measure cannot be a computable real
- The low and c.e. paths need not be the same (Arslanov 1981)
- The minimal pair need not both be Δ^0_2 (Kučera 1988)

PA, and hence DNR_2, satisfies all but the second of these.

Consequences of antibasis theorems

We can't *a priori* say anything about separating sets, since not all Π^0_1 classes are separating classes (though many antibasis theorems hold for separating classes as well – as DNR_2 witnesses).

However, we can say there is a computable function with no computable zeros, even one that has a set of zeros of positive measure but still no computable one.

More membership theorems

We have a lot of degree control (Jockusch and Soare, 1972):

- There is a nonempty Π^0_1 class such that the only c.e. degree $\geq_T$ any path of the class is $\mathbf{0'}$.
- For any c.e. degree $\boldsymbol{c}$ there is a Π^0_1 class such that the degrees of its c.e. paths are exactly those $\geq_T \boldsymbol{c}$.
- For any degree $\boldsymbol{a}$ there is a nonempty Π^0_1 class with no members of degree $\mathbf{0}$ or $\boldsymbol{a}$.
- There is a nonempty Π^0_1 class all of whose members are Turing incomparable.

A version of that last one for separating classes (JS '72):

General: There is a nonempty Π^0_1 class all of whose members are Turing incomparable.

Specific: There are disjoint c.e. sets A and B that are computably inseparable such that any two separating sets of A and B either have finite difference or are Turing incomparable.

Connections to randomness

We take on faith that the random reals are exactly those that pass the *universal Martin-Löf test*. That is, there is a computable sequence of Σ^0_1 classes (subsets of 2^ω) such that the nonrandom reals are exactly those reals in the intersection of the sequence. Furthermore the n^{th} class in the sequence has measure bounded by 2^{-n} (Denis will elaborate).

As the complement of a Σ^0_1 class is a Π^0_1 class, there are Π^0_1 classes all of whose elements are random; in fact with measure arbitrarily close to 1.

The Π^0_1 classes of positive measure are exactly those containing a random real (observation/Kurtz).

Every Π^0_1 class of positive measure has an element of every 1-random degree (Kučera).

Downey and Miller jump inversion (2006):

If P is a Π^0_1 class of positive measure, then for every Σ^0_2 set $S \geq_T 0'$, there is a Δ^0_2 real $A \in P$ such that $A' \equiv_T S$.

Taking P to be one of the Π^0_1 classes containing only random reals, we get a Δ^0_2 random real A that jumps to S.

Enumerations

Before constructing an enumeration of all Π^0_1 classes, we show the complexity of tree representation is flexible:

Proposition. For any $P \subseteq 2^\omega$, TFAE:

(a) $P = [T]$ for some Π^0_1 tree $T \subseteq 2^{<\omega}$;

(b) $P = [T]$ for some computable tree $T \subseteq 2^{<\omega}$;

(c) $P = [T]$ for some primitive recursive tree $T \subseteq 2^{<\omega}$.

Another proof:

(c) $\Rightarrow$ (b) $\Rightarrow$ (a) is clear.

(a) $\Rightarrow$ (b): From Π^0_1 T given by computable relation R such that $\sigma \in T \Leftrightarrow (\forall n)R(n, \sigma)$, build computable tree $S \supseteq T$:

$$\sigma \in S \iff (\forall m, n \leq |\sigma|)R(m, \sigma \upharpoonright n).$$

(b) $\Rightarrow$ (c): From computable T given by total computable $\{0, 1\}$-valued function φ such that $\sigma \in T \Leftrightarrow \varphi(\sigma) = 1$, build primitive recursive tree $S \subseteq T$:

$$\sigma \in S \iff (\forall n < |\sigma|)\neg\varphi_{|\sigma|}(\sigma \upharpoonright n) = 0.$$

Enumerating the Π^0_1 classes via primitive recursive trees

For $\{W_e\}_{e\in\omega}$ an enumeration of all c.e. sets and $\{\sigma_e\}_{e\in\omega}$ an enumeration of $2^{<\omega}$ (lexicographically, say), define the tree T_e by

$$\sigma \in T_e \Leftrightarrow (\forall n < |\sigma|) \left[\sigma_n \subseteq \sigma \rightarrow n \notin W_{e,|\sigma|}\right].$$

Then $P_e = [T_e]$ enumerates all Π^0_1 classes.

Note that neither the proposition about equivalence of representations nor the construction of the enumeration of Π^0_1 classes is dependent on using Π^0_1 subclasses of 2^ω; both will go through if we use ω^ω. We will stick to the former but there are many additional index set results for ω^ω.

Recall that given an enumeration $\{\xi_e\}_{e\in\omega}$ (of anything) an *index set* $\mathcal{I}$ is any subset of ω such that if $a \in \mathcal{I}$ and $\xi_a = \xi_b$, then $b \in \mathcal{I}$.

A set $A \subseteq \omega$ is H^m_n-*complete* (for $H = \Pi, \Sigma, \Delta$) if it is H^m_n and every other H^m_n set B is 1-reducible to A.

In our setting, the index sets will often be properties of trees, but sets of indices of Π^0_1 classes. That is, many of the sets will be of the form

$$\mathcal{I} = \{e : P_e \text{ has a tree representation with property } \alpha\},$$

and all indices i of P_e will be in the set if at least one of them corresponds to a tree T_i with property α.

Why should we care?

We can transfer these results to statements about index sets of computable mathematical problems. For instance, the index set of primitive recursive graphs with a 4-coloring is Π^0_1-complete, but the index set of those with a *computable* 4-coloring is Σ^0_3-complete; this strengthens the result that there is a computable 4-colorable graph with no computable 4-coloring.

Let $\mathcal{I}(\mathcal{P})$ be the index set of classes with property $\mathcal{P}$.

- $\mathcal{I}$(nonempty) is Π^0_1-complete.
- $\mathcal{I}$(no more than c paths) is Π^0_2-complete for fixed $c \geq 1$.
- $\mathcal{I}$(exactly c paths) is Π^0_2-complete for $c = 1$ and D^0_2-complete for $c > 1$.
- $\mathcal{I}$(finite) is Σ^0_3-complete.
- $\mathcal{I}$(countable) is Π^1_1-complete.

D^m_n sets are those expressible as the difference of two Σ^m_n sets.

A few more, as we are often interested in the existence of computable solutions to problems:

- $\mathcal{I}$(no comp. paths) and $\mathcal{I}$(nonempty; no comp. paths) are Σ^0_3-complete.
- $\mathcal{I}$(more than c comp. paths) is Σ^0_3-complete.
- $\mathcal{I}$(exactly c comp. paths) is D^0_3-complete.
- $\mathcal{I}$(infinitely many comp. paths) is Π^0_4-complete.

Again: D^m_n = difference of two Σ^m_n sets.

Cenzer and Remmel (CR):

There exist computable functions taking indices for computably continuous functions (CCFs) on 2^ω to indices for Π^0_1 classes representing their set of zeroes and conversely.

This allows us to transfer index set results. For example,

- The index set of CCFs which have exactly c zeros for any fixed $c \geq 1$ is D^0_2-complete.
- The index set of CCFs which have exactly c computable zeros for any fixed $c \geq 1$ is D^0_3-complete.
- The index set of CCFs which have more than c zeros for any fixed $c \geq 1$ is Σ^0_2-complete.
- The index set of CCFs which have more than c computable zeros for any fixed $c \geq 1$ is Π^0_3-complete.

One more theorem about tree representations

In fact, polynomial-time computable trees suffice to represent all Π^0_1 classes.

Of course, must say what we mean by polynomial-time computable tree. In 2^ω it is straightforward; if we were dealing with Π^0_1 classes in a different space we would have to do some work.

Given a computable function φ for a tree T, we approximate T by T_s, where

$$\sigma \in T_s \Leftrightarrow \varphi_s(\sigma)\uparrow \text{ or } \downarrow = 1.$$

The p-time tree P is defined by

$$\sigma \in P \Leftrightarrow (\forall \tau \subset \sigma)[\tau \in T_{|\sigma|}].$$

Lattice Structure

The collection of all Π^0_1 classes ordered by inclusion forms a distributive lattice, denoted $\mathcal{E}_\Pi$.

Top and bottom: 2^ω and $\emptyset$

Meet and join: $\cap$ and $\cup$

Atoms (minimal elements): singletons (computable paths)

Complemented elements: clopen sets (finite unions of intervals)

Intervals in the lattice: $[P, P'] = \{Q \in \mathcal{E}_\Pi\colon P \subseteq Q \subseteq P'\}$

Once we have a lattice, we can look at intervals of and embeddings into the lattice, as well as definability. There are several computably isomorphic (though order-reversing) settings we can work in to obtain these results.

- $\mathcal{E}_{\Pi}$ itself;
- the lattice of c.e. ideals/filters of the countable atomless Boolean algebra Q;
- the lattice of c.e. ideals/filters of $2^{<\omega}$;

[Isomorphisms laid out in CCDH and W]

The isomorphic setting we will use is the c.e. ideals of $2^{<\omega}$:

A string $\sigma \in 2^{<\omega}$ is a *nonextendible node* of the Π^0_1 class P if $[\sigma] \cap P = \emptyset$.

If $[T] = P$ for a computable tree T, $\sigma \notin T$ is nonextendible, and $\sigma \in T$ such that all extensions of σ dead-end is also nonextendible.

The nonextendible nodes of P for any Π^0_1 P form a c.e. ideal of $2^{<\omega}$; can see the isomorphism is order-reversing.

Intervals

There are exactly two isomorphism types for nontrivial end segments of $\mathcal{E}_\Pi$.

– Cholak, Coles, Downey, Herrmann (CCDH):

If $P \subsetneq 2^\omega$ is a clopen Π^0_1 class, then $[P, 2^\omega] \cong \mathcal{E}_\Pi$ computably.

If $P, Q \in \mathcal{E}_\Pi$ are nonclopen, then $[P, 2^\omega] \cong [Q, 2^\omega]$ computably.

– Cenzer and Nies (CN2):

If $P \in \mathcal{E}_\Pi$ is nonclopen, then $[P, 2^\omega] \ncong \mathcal{E}_\Pi$.

The computable isomorphisms are easiest to see in the setting of c.e. ideals, where we are looking at an *initial* segment (interval of all ideals contained in the given ideal).

The *root set* of an ideal $\mathcal{I}$ is the minimal generating set: $\{\sigma_i\}_{i\in I}$ such that $\mathcal{I} = \{\tau : (\exists i \in I)(\tau \supseteq \sigma_i)\}$ and $i \neq j \Rightarrow \sigma_i \perp \sigma_j$.

A clopen Π^0_1 class corresponds to an ideal with a finite root set of size $k+1$, say; we may map the i^{th} element to 1^i0, $0 \leq i < k$, with the final element mapping to 1^k. (If the root set has size 1 map it to the empty node.) Fill in $2^{<\omega}$ in the natural way; this generates an isomorphism on ideals.

A *basis* of an ideal is a set B that generates the ideal such that any two elements of B are incomparable.

For the nonclopen isomorphism we need a lemma:

Any c.e. ideal has a c.e. basis.

Given a nonclopen Π^0_1 class, let $\{\sigma_i\}_{i\in\omega}$ be a c.e. basis for the associated c.e. ideal. We'll map it to one standard nonclopen ideal: the one with root set $\{1^j 0 : j \in \omega\}$. Map the basis to the root set (in order of enumeration) and fill in $2^{<\omega}$ in the natural way; the map generated is an isomorphism between the two initial segments of ideals.

The nonisomorphism between end segments starting with clopen or nonclopen Π^0_1 classes is harder to prove.

(CN2) is a contradiction argument. Nies later found a Σ^0_3-definable difference in the setting of c.e. ideals of the countable atomless Boolean algebra Q.

For two ideals $A, E \in I(Q)$, A is *small in* E ($A \subset_s E$) if $A \subset E$, E is noncomplemented in $I(Q)$, A is noncomplemented in $[0, E]$, and if $Y \subseteq A$ is complemented in $[0, E]$, then Y is also complemented in $I(Q)$.

Let β be the statement $\exists E\, \exists A\, (A \subset_s E)$. $I(Q) \models \beta$ but for nonprincipal (corresponding to nonclopen Π^0_1 class) ideal M, $[0, M] \not\models \beta$.

Thin classes

A Π^0_1 class P is *thin* if every Π^0_1 subclass of P is relatively clopen; that is, for each $Q \subseteq P$ there is clopen $C \subseteq 2^\omega$ such that $Q = P \cap C$.

The thin Π^0_1 classes are exactly those P such that $[\emptyset, P]$ is a Boolean algebra (i.e. distributive, complemented lattice) - hence thinness (including finite) is definable in $\mathcal{E}_\Pi$.

The index set of thin classes is Π^0_4-complete.

Perfect thin classes

A Π^0_1 class is *perfect* if it has no isolated paths. In other words, every extendible node of its representative tree has at least two incomparable extensions.

Perfect thin classes P are exactly those such that $[0, P]$ is an atomless Boolean algebra, because a computable path must be isolated in a thin class; hence they are definable in $\mathcal{E}_\Pi$.

CCDH show that perfect thin classes witness degree invariance of the array noncomputable degrees. That is, each anc degree is represented by a perfect thin class and all perfect thin classes have anc degree; definability implies invariance under automorphisms. In fact they form an orbit.

Minimal classes

A Π^0_1 class P is *minimal* if every Π^0_1 subclass of P is finite or cofinite in P.

The minimal Π^0_1 classes are the atoms when working with $\mathcal{E}^*_\Pi$ ($:= \mathcal{E}_\Pi$ modulo finite difference).

The minimal classes with noncomputable paths are exactly the thin classes with exactly one non-isolated point.

The index set of minimal classes is Π^0_4-complete.

Comparisons between $\mathcal{E}_\Pi$ and $\mathcal{E}$, the lattice of c.e. sets, are often fruitful lines of research.

Nies proved that if an interval of $\mathcal{E}$ is not a Boolean algebra, then it has an undecidable theory (in fact its theory interprets true arithmetic).

Cenzer and Nies (CN1) proved that there are intervals of $\mathcal{E}_\Pi$ that are not Boolean algebras but have decidable theories.

The proof is via $\mathcal{E}_\Pi^*$.

Given a lattice $(L, \leq)$ we denote join (l.u.b.) by $\vee$ and meet (g.l.b.) by $\wedge$; the greatest and least elements are 1 and 0.

$(L, \leq)$ satisfies the *dual reduction property* if for any $a, b \in L$, there exist $a_1 \geq a$ and $b_1 \geq b$ such that $a_1 \vee b_1 = 1$ and $a_1 \wedge b_1 = a \wedge b$.

(CN1) step 1: For any finite distributive lattice L that satisfies the dual reduction property, there is a Π^0_1 class P such that $[\emptyset, P]^* \cong L$.

Small pieces of proof:

For one-element L, any finite P will do.

For two-element L, P must be minimal, so $[\emptyset, P]^*$ will have two elements.

For larger L, the construction of a minimal Π^0_1 class is generalized: make P with subclasses that are "minimal over" each other (aligned with the structure of L).

(CN1) step 2: For the P constructed, $[\emptyset, P]$ is isomorphic to a sublattice of the $\mathcal{P}(\mathbb{N})$ that is closed under finite differences.

Lachlan: If a lattice $L \subset \mathcal{P}(\mathbb{N})$ is closed under finite differences, then the theory of L is many-one reducible to the theory of L^*.

(CN1) step 3: The theory of $[\emptyset, P]$ is many-one reducible to the theory of L (the original finite lattice) and is hence decidable.

So in $\mathcal{E}$, "not a Boolean algebra" $\Rightarrow$ "interprets true arithmetic".

In $\mathcal{E}_\Pi$, "not a Boolean algebra" doesn't even imply "undecidable".

However, if $P \in \mathcal{E}_\Pi$ is *decidable* and $[\emptyset, P]$ is not a Boolean algebra, then the theory of $[\emptyset, P]$ interprets true arithmetic.

Decidability for a Π^0_1 class P means the tree T with no dead ends such that $[T] = P$ is computable.

Some topics we didn't cover

- Π^0_1 classes in ω^ω, $\mathbb{R}$, or $[0, 1]$
- The Cantor-Bendixson derivative and rank
- Reverse mathematics and Ramsey theory
- The structure of the lattice $[P, 2^\omega]$ for P nonclopen
- More examples and applications: graph theory and combinatorics, orderings, nonmonotonic logic

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