DEGREE INVARIANCE IN THE Π^0_1 CLASSES

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Abstract. Let $\mathcal{E}_\Pi$ denote the collection of all Π^0_1 classes, ordered by inclusion. A collection of Turing degrees $\mathcal{C}$ is called *invariant* over $\mathcal{E}_\Pi$ if there is some collection $\mathcal{S}$ of Π^0_1 classes representing exactly the degrees in $\mathcal{C}$ such that $\mathcal{S}$ is invariant under automorphisms of $\mathcal{E}_\Pi$. Herein we expand the known degree invariant classes of $\mathcal{E}_\Pi$, previously including only $\{\mathbf{0}\}$ and the array noncomputable degrees, to include all high$_n$ and non-low$_n$ degrees for $n \geq 2$. This is a corollary to a very general definability result. The result is carried out in a substructure G of $\mathcal{E}_\Pi$, within which the techniques used model those used by Cholak and Harrington [6] to obtain the same definability for the c.e. sets. We work back and forth between G and $\mathcal{E}_\Pi$ to show that this definability in G gives the desired degree invariance over $\mathcal{E}_\Pi$.

1. Introduction

A Π^0_1 class is the collection of infinite paths through a computable subtree of the complete binary-branching tree, $2^{<\omega}$. These classes became important in computability theory initially because one may encode many structures into the complete binary-branching tree in such a way that the paths of a Π^0_1 class encode all examples of a specific substructure, such as prime ideals of a c.e. commutative ring. By proving results about the paths of Π^0_1 classes in general one may draw conclusions about, e.g., the degrees of these substructures. As a result, the collection of Π^0_1 classes as a whole, called $\mathcal{E}_\Pi$ after $\mathcal{E}$, the lattice of c.e. sets, has also become the object of much study. For general background on Π^0_1 classes and $\mathcal{E}_\Pi$, see [1, 2, 4].

A collection of Turing degrees is called *invariant* over a structure P if it corresponds to a subset of P that is closed under isomorphisms. Degree invariance argues for the naturality of a collection of degrees with regard to P. For example, $\{\mathbf{0}\}$ is degree invariant over $\mathcal{E}$ by correspondence to the finite sets and $\{\mathbf{0'}\}$ by the creative sets (Harrington; see Soare §XV.1 [12]). Invariance of the high$_1$ and non-low$_2$

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degrees, proved by Martin [10] and Lachlan and Schoenfield [8, 11], respectively, led Martin to conjecture that the nontrivial degree invariant classes over $\mathcal{E}$ were the high_{2n-1} and non-low_{2n} classes; this was later refined to conjecture these were exactly the nontrivial invariants from among the jump classes (non-)high_n and (non-)low_n (see Harrington and Soare [7]).

Cholak and Harrington [6] resolved Martin's conjecture negatively with a sweeping definability result that gives, as a corollary, the degree invariance over $\mathcal{E}$ of the classes high_n and non-low_n for *all* $n \geq 2$. In this paper we prove the Cholak and Harrington result in a setting that gives the same degree invariance results for $\mathcal{E}_\Pi$; in particular, we have the following theorem.

Theorem 8.3. *For all $n \geq 2$, the* high_n *and* non-low_n *c.e. degrees are invariant over $\mathcal{E}_\Pi$.*

This greatly expands the known invariant classes of $\mathcal{E}_\Pi$, as the only other examples are $\{\mathbf{0}\}$ and the collection of array noncomputable (anc) degrees, the latter shown by Cholak, Coles, Downey, and Herrmann [5] via perfect thin Π^0_1 classes. Note that though the present result is for $\mathcal{E}_\Pi$, the techniques are squarely in the realm of $\mathcal{E}$.

The proof is carried out in G, an end segment of the Π^0_1 classes containing all classes which include a particular non-clopen subclass. Usually that class is a singleton f, so the elements of G are exactly the Π^0_1 classes containing f. We will show in §3.1 that degree invariance in G gives degree invariance in $\mathcal{E}_\Pi$. It appears that using a restricted structure such as G is necessary so that finitely-many stages of construction are guaranteed not to use up all the room to work. In $\mathcal{E}_\Pi$ as a whole, we could eliminate all paths in the tree by truncating only the strings 0 and 1, which would be analogous to enumerating all of ω in two stages; in G truncating finitely-many nodes always leaves infinitely-many paths in the tree.

The motivation for looking at G stems from our earlier paper [13], in which the quotient structure $G^\diamondsuit$ of G is proved isomorphic to $\mathcal{E}^*$, the c.e. sets modulo finite difference. The equivalence relation $=^\diamondsuit$ which gives rise to $G^\diamondsuit$ is "eventual equality"; if f is the computable singleton included in all elements of G, then $A =^\diamondsuit B$ if and only if for some n, $A \cap [f \upharpoonright n] = B \cap [f \upharpoonright n]$. One might hope the present result would be a corollary to that isomorphism, but the map does not preserve Turing degree, and in fact Turing degree is not well-defined in $G^\diamondsuit$ ([13], Claim 6.3). However, we will still use the $=^\diamondsuit$ perspective extensively and explicitly; as will be explained in §4, we need an analogue to working modulo finite difference. In $\mathcal{E}$ the close relationship between sets that are $*$-equal is essential for constructions with injury to work. The relationship between Π^0_1 classes that are $\diamondsuit$-equal is not as close (in particular they need not be Turing equivalent), but we will show we still can succeed working only up to a principal ideal.

This paper is organized as follows: §2 contains the basic definitions and conventions, and §3 discusses definability and invariance. §4 defines $=^\diamondsuit$ and related notions, including new versions of the Friedberg and Owings Splitting Theorems. §§5–8 give the definitions and proofs leading to the main result, and §9 makes some comments on the general proof scheme and surveys open questions.

Notation for functions and sets, and computability-theoretic terminology, will follow Soare [12].

2. Ideals and G

Although our results will be for $\mathcal{E}_\Pi$, we will actually work in one of two isomorphic structures, $I(Q)$ or $I(2^{<\omega})$, of c.e. ideals. As shown in Cholak, Coles, Downey, and Herrmann [5] and our [13], these three structures are computably isomorphic in a natural way, though when one moves from Π^0_1 classes to ideals or vice-versa, order is reversed.

$I(Q)$ is the lattice of c.e. ideals of the countable atomless Boolean algebra, Q. One may view Q as a collection of propositional formulas modulo tautological equivalence and ordered by logical implication, where the independent elements $\{p_i : i \in \omega\}$ generate Q. An ideal of Q is a subset I closed under disjunction and downward under implication (if $\sigma \in I$ and $\tau \to \sigma$, then $\tau \in I$). The least element of the lattice $I(Q)$ is denoted 0, and is the equivalence class of logically contradictory formulas; the greatest element is Q.

$I(2^{<\omega})$ is the lattice of c.e. ideals of $2^{<\omega}$, the set of all finite binary strings. The notation we use is standard, but we highlight a few: $|\sigma|$ is the length of σ, $\sigma \subseteq \tau$ means τ extends or equals σ, and $\tau \upharpoonright i$ is the *initial segment of τ of length i*; that is, the unique string $\sigma \in 2^{<\omega}$ of length i such that $\sigma \subseteq \tau$. The concatenation of strings σ and τ will be denoted $\sigma^\frown\tau$ or just $\sigma\tau$. In $2^{<\omega}$ an ideal is a set closed under extension (if $\sigma \in I$, every $\tau \supset \sigma$ is in I) and meet (if both $\sigma^\frown 0$ and $\sigma^\frown 1$ are in I, so is σ). The least element is $\emptyset$ and the greatest is $2^{<\omega}$.

In both settings, the ideal *generated* by a set X, denoted $\langle X\rangle$, is the closure of X under the appropriate operations.[1] A finitely-generated ideal is called *principal*; in Q this is equivalent to generation by a single element but in $2^{<\omega}$ it is not. To distinguish, in $I(2^{<\omega})$ we call the ideal

[1]This notation is identical to that which will be used for the pairing function, but which is intended should be clear in context.

generated by a single string σ the *cone* or *interval* above σ.[2] In both structures least upper bound (*join*) is $X \vee Y = \langle X \cup Y \rangle$, and greatest lower bound (*meet*) is intersection.

An ideal is c.e. if it is computably enumerable as a set. Principal ideals are computable and we refer to them without using an enumeration. By convention, we will enumerate nonprincipal ideals as nested sequences of principal ideals: the *generating sequence* $\{a_s : s \in \omega\}$ for the ideal A gives stagewise approximations $A_s := \langle a_t : t \leq s \rangle$. We will always assume our generating sequences are pairwise disjoint, meaning for $s > 0$, $\langle a_s \rangle \cap A_{s-1} = 0$ or $\emptyset$ (as appropriate). Any c.e. generating sequence may be computably converted into a pairwise disjoint c.e. generating sequence, in either $I(Q)$ or $I(2^{<\omega})$.

In $X = I(Q)$ or $I(2^{<\omega})$, we make the following definition.

Definition 2.1. Fix $M \in X$ nonprincipal. G is the initial segment $\{I : I \subseteq M\}$.

All copies of G are computably isomorphic (Cholak, Coles, Downey, and Herrmann [5]),[3] so this is well-defined; when we must distinguish which M is the top element we will write G_M. For purposes of definability we will restrict to M maximal.

2.1. **Terminology and notation that bears highlighting.** We fix an enumeration of all ideals of G (i.e., c.e. subideals of M) and denote it $\{I_e : e \in \omega\}$. We also fix an enumeration $\{m_0, m_1, \ldots\}$ of all elements of M (not just a generating set), and let $x \lhd y$ indicate x is enumerated before y. Let $P_{\unlhd x}$ be the principal ideal generated by all elements of M enumerated up to and including x, and $P_{\lhd x}$ the ideal generated by all elements of M enumerated up to but not including x. When we know which $m_i \in M$ we are working with, we have the shorthand $P_{<i} := P_{\lhd m_i}$ and $P_{\leq i} := P_{\unlhd m_i}$.

In $I(2^{<\omega})$ we will refer to the *cones off* an infinite sequence $x \in 2^\omega$. These are all the cones $\langle \sigma \rangle$ such that σ without its last bit is an initial segment of x but $\sigma \not\subset x$. For example, the cones off 0^ω are the cones $\langle 0^n 1 \rangle$ for all $n \in \omega$. Also in $I(2^{<\omega})$ we will make extensive use of $G_{M_0} \subset I(2^{<\omega})$, where $M_0 = 2^{<\omega} - \{0^n : n \in \omega\}$.

Recall for c.e. sets X, Y with enumerations $\{X_s\}$, $\{Y_s\}$ we define

$$X \setminus Y = \{x : \exists s (x \in X_s \,\&\, x \notin Y_s)\},$$

[2]This too is a slight abuse, as usually the cone or interval generated by σ refers to the collection of all infinite sequences that extend σ.

[3]If M is principal, the initial segment is computably isomorphic to $I(Q)$ [5]; Cenzer and Nies showed $G \not\cong I(Q)$ [3].

$X \searrow Y = (X \setminus Y) \cap Y$, and $X - Y = X \cap \overline{Y}$. The last need not be a c.e. set; if we try to make an analogous definition for ideals we find it not only need not be a c.e. ideal, it need not be an ideal at all. The intersection of X with the set-theoretic complement of Y (a direct lift of the $\mathcal{E}$ definition) is rarely an ideal, and the ideal complement of Y (toward an analogous definition) need not exist.

We will, however, use the $-$ notation to mean set-theoretic difference, with the understanding that the set of elements so obtained is neither c.e. nor an ideal in general. However, the notation $\overline{X}$ will mean the ideal complement of X; there is only one place where we want a set-theoretic complement and we will use $-$ for it.

The other two subtraction operations have sensible ideal versions and are useful. $X \setminus Y$ is the ideal

$$\{x : \exists s(x \in X_s \;\&\; \langle x\rangle \cap Y_s = 0)\},$$

and $X \searrow Y$ is $(X \setminus Y) \cap Y$. For an ideal Z, if there are two ideals which meet to 0 and join to Z we will call them a *split* of Z; note that by staggering our enumerations we can view $X \searrow Y$ and $Y \searrow X$ as a split of $X \cap Y$, though it need not be nontrivial.

3. Definability and invariance

The main result herein has to do with definability, and as a corollary invariance. For our purposes there are two kinds of each, beginning with *set invariance* and *degree invariance*. Given a structure P, the collection $\mathcal{S} \subseteq P$ is set invariant if it is closed under automorphisms of P. We will use the term "set invariant" regardless of whether the elements of P are actually sets. When the elements of P have Turing degree, we make the following definition.

Definition 3.1. A collection of degrees $\mathcal{C}$ is *(degree) invariant* in P if there is $\mathcal{S} \subseteq P$ such that

(i) For every degree $\boldsymbol{d} \in \mathcal{C}$, there is $X \in \mathcal{S}$ of degree $\boldsymbol{d}$,
(ii) If $X \in \mathcal{S}$ has degree $\boldsymbol{d}$, then $\boldsymbol{d} \in \mathcal{C}$, and
(iii) $\mathcal{S}$ is set invariant.

Invariance is one way to argue for the naturality of a property of degrees; if the collection of degrees with that property corresponds to an invariant collection of elements of P, one could argue this indicates the property is more "internal" to P than one which corresponds to no such collection.

The two kinds of definability at work here are in the language of arithmetic and of inclusion, the latter the order relation for all of our structures (and the only relation automorphisms must preserve).

The first will be called *arithmetic definability*, and is used for degree-theoretic work: we are concerned with which level of the arithmetic hierarchy our formulas occupy so that our result can relate to the double jumps of c.e. ideals. The second is *definability in the language of inclusion*, or $\{\subseteq\}$-*definability* for short; since the only relation in any of our structures P is $\subseteq$, this kind of definability implies set invariance in P. Here, the quantifier depth of formulas (or even that formulas are finitary) is not essential information; we need merely $\mathcal{L}_{\omega_1,\omega}$-definability in $\{\subseteq\}$. In Cholak and Harrington's double jump paper [6], the distinction is not always sharply drawn, but here, with a less well-studied structure at hand, we will be careful to specify which we mean.

3.1. **Pushing invariance up.** To obtain degree invariance results for $\mathcal{E}_\Pi$ via G we must ensure first that the Cholak–Harrington result holds in G, and second that it transfers to $I(Q)$ and hence $\mathcal{E}_\Pi$.

The first step involves both arithmetic and $\{\subseteq\}$-definability. Checking arithmetic complexity will happen along the way, as definitions are made. The primary relation for which $\{\subseteq\}$-definability is needed is $\subseteq^\diamondsuit$. It is clear that $\subseteq^\diamondsuit$ is definable in $I(Q)$, because there being principal is equivalent to being complemented (see [5]):

$$A \subseteq^\diamondsuit B \;\Leftrightarrow\; (\exists P)[P \text{ principal } \& \; A \vee P \subseteq B \vee P].$$

In $I(Q)$ we may also definably work within a fixed copy of G, with a parameter for its maximal element M. It is not clear, however, that this can be pushed down to G; in fact it is open whether being principal is finitarily definable in G. Fortunately we don't actually care whether $\subseteq^\diamondsuit$ is $\{\subseteq\}$-definable in G, but only that it is invariant under automorphisms of G. The following result shows that definability in $I(Q)$ is sufficient for that invariance (and that being principal is $\mathcal{L}_{\omega_1,\omega}$ definable in G). Hence, our $\{\subseteq\}$-definability work will be in $I(Q)$ with a parameter for M, though that will be entirely in the background for the remainder of the paper.

Lemma 3.2. *Any property $\{\subseteq\}$-definable in $I(Q)$ (possibly with parameter M) is invariant in G.*

Proof. Let P be a property $\{\subseteq\}$-definable in $I(Q)$ (with parameter M) and let $I \in G$ be an ideal with property P. Let $\Phi : G \to G$ be an automorphism; Φ extends to an M-preserving automorphism $\Psi : I(Q) \to I(Q)$ such that $\Psi \restriction G = \Phi$ ([13], Claim 5.2). The image $\Psi(I) = \Phi(I)$ must have property P by definability in $I(Q)$, and hence P is preserved by Φ. □

We next show that it does not matter which copy of G we work in provided its maximal element is a maximal ideal of $I(Q)$. The following result is used in [13] but not explicitly shown. It allows us to take our invariant collection of ideals in G and unambiguously refer to its "isomorphic copy" in other copies of G, since invariant classes are unions of orbits.

Lemma 3.3. *Given M_1, M_2 maximal c.e. subideals of Q, an orbit U_{M_1} of G_{M_1}, and isomorphisms $\Phi, \Psi : G_{M_1} \to G_{M_2}$, the images $\Phi(U_{M_1})$ and $\Psi(U_{M_1})$ are equal.*

Proof. To show $\Phi(U_{M_1})$ is closed, suppose $A \in \Phi(U_{M_1})$ and $B = \Theta(A)$ for some automorphism Θ of G_{M_2}. Then $\Phi^{-1} \circ \Theta \circ \Phi$ takes $\Phi^{-1}(A)$ to $\Phi^{-1}(B)$, showing $B \in \Phi(U_{M_1})$. Likewise, $\Psi(U_{M_1})$ is closed under automorphisms of G_{M_2}.

Now let $A \in \Phi(U_{M_1})$ and $B \in \Psi(U_{M_1})$. Then $\Phi^{-1}(A)$ is automorphic to $\Psi^{-1}(B)$ via some automorphism $\Theta : G_{M_1} \to G_{M_1}$, so A and B are automorphic in G_{M_2} by $\Psi^{-1} \circ \Theta \circ \Phi$. This shows that any element of $\Psi(U_{M_1})$ is automorphic to any element of $\Psi(U_{M_1})$, and in particular (letting $\Psi = \Phi$) that each image is transitive. Since each image is closed, they are orbits and must be equal. □

We will denote the isomorphic copy of U_{M_1} in G_{M_2} by U_{M_2}. We use the same subscript notation for invariant classes to indicate in which copy of G we are working. As shown in [13], taking the union of isomorphic copies of an orbit in G_M over all maximal M gives an orbit in $I(Q)$ of the same arithmetic complexity. Taking the union of invariant classes hence gives an invariant class; it is this union that will give the degree invariance result for $I(Q)$. There is one more component to the proof, however, which follows from automorphism results of Cholak, Coles, Downey, and Herrmann [5].

Lemma 3.4. *For a fixed orbit U and any maximal M_1, M_2, the sets $\{\deg I : I \in U_{M_1}\}$ and $\{\deg I : I \in U_{M_2}\}$ are equal.*

Proof. Given $I \in U_{M_1}$, we show there is some $J \in U_{M_2}$ with $J \equiv_T I$. Choose a computable automorphism Φ of $I(Q)$ taking M_1 to M_2. Φ is induced by a unique automorphism of Q ([5], Theorem 6.1) which is also computable ([5], Theorem 6.4). Hence $J = \Phi(I)$ has the same Turing degree as I, and by Lemma 3.3 J must be in U_{M_2}. □

Corollary 3.5. *If a collection $\mathcal{C}$ of c.e. degrees is invariant over G via $\mathcal{S}$, it is invariant over $I(Q)$ via the union of $\mathcal{S}_M \subseteq G_M$ over all maximal ideals $M \in I(Q)$.*

Corollary 3.6. *If a collection $\mathcal{C}$ of c.e. degrees is invariant over G, it is invariant over $\mathcal{E}_\Pi$.*

4. Adding diamonds to everything

The main technical difficulty in moving results to G is that although *principle* is analogous to *finite*, it is far from equivalent. In $\mathcal{E}$, constructions are typically carried out modulo finite difference, at least implicitly. The fact that $A =^* B$ implies both $A \equiv_T B$ and that A is c.e. if and only if B is c.e. allows constructions to succeed despite injury. For example, if we wish to construct a split of C, it is enough to build A and B such that

$$A \cup B = C \text{ and } A \cap B =^* \emptyset, \tag{4.1}$$

though the definition of split has $A \cap B = \emptyset$. In general we do not make any special note of this. It is natural (and necessary) to make the same allowance in G, constructing a split of Z via ensuring

$$X \vee Y = Z \text{ and } X \cap Y \text{ principal}, \tag{4.2}$$

but we must be explicit about it. In the c.e. sets, the existence of B satisfying (4.1) implies the existence of some $\hat{B} =^* B$ satisfying $A \cup \hat{B} = C$ and $A \cap \hat{B} = \emptyset$, where $\hat{B} \equiv_T B$ is also c.e. In the c.e. ideals, the existence of Y satisfying (4.2) implies nothing. Not only need there not be a c.e. ideal $\hat{Y} \equiv_T Y$ giving $X \vee \hat{Y} = Z$ and $X \cap \hat{Y} = 0$, there need not be any ideal at all that gives a true split of Z with X.

For example, in G_{M_0} let $I = \langle 1^n 0, 0^{2n} 1 : n \geq 1\rangle$; it is all but the first cone off 1^ω and every other later cone off 0^ω (see §2.1). It is clearly complemented outside $\langle 1\rangle$ by $\langle 0^{2n+1}1 : n \geq 0\rangle$, the remaining cones off 0^ω. However, if $\hat{I}$ is to be the complement of I, $I \vee \hat{I} = M_0$ must contain in particular $\{1^n : n \geq 1\}$, so $\hat{I}$ must contain some 1^m. Hence all cones $\langle 1^n 0\rangle$ for $n \geq m$ are in $\hat{I}$, giving $I \cap \hat{I} \neq \emptyset$.

Logistically, making the mod-principal viewpoint explicit usually adds an extra index to requirements, labeling the principal ideal at hand. Globally it means we must prove additional "behind the scenes" theorems, as well as ensure that arithmetic complexity does not increase and invariance under automorphisms still holds, as discussed in §3.1.

We define the equivalence relation $=^\diamond$ on G by

$$A =^\diamond B \iff (\exists m \in M)[A \vee \langle m\rangle = B \vee \langle m\rangle].$$

In other words, $A =^\diamond B$ when their differences are contained in a principal subideal of M. We say m or $\langle m\rangle$ *witnesses* $A =^\diamond B$, and likewise for other mod-principal relations. We also define

$$A \subseteq^{\diamondsuit} B \iff (\exists m \in M)[A \subseteq B \vee \langle m \rangle].$$

As shown in [13], $\subseteq^{\diamondsuit}$ is Σ^0_3 in the language of arithmetic, just as $\subseteq^*$ is. The quotient structure $G/{=^{\diamondsuit}}$ is denoted $G^{\diamondsuit}$.

Finally, we make a definition to be used solely for arithmetic definability, analogous to $(\exists^\infty x) \equiv (\forall n)(\exists x > n)$ for ω.

Definition 4.1. Let $(\forall m \in M)(\exists x \in M)[x \notin \langle m \rangle \ \& \ \varphi(x)]$ abbreviate $(\exists^{np} x)[\varphi(x)]$. Verbally this will be described as a *nonprincipal collection*; a set which may not itself be an ideal, but which cannot be contained in any principal ideal.

Since M is maximal, and membership in a maximal or principal ideal is computable, there is no arithmetic complexity increase over $(\exists^\infty x)$. We may also refer to a non-ideal set contained in a principal ideal as a *principal collection*.

4.1. **$\diamondsuit$-complementation.** The mod-principal example at the start of the section is $\diamondsuit$-splitting. For $X \subseteq Y$ c.e. ideals, X is a *$\diamondsuit$-split* of Y, denoted $X \sqsubseteq^{\diamondsuit} Y$, if there is some c.e. $Z \subseteq Y$ such that $X \cap Z =^{\diamondsuit} 0$ and $X \vee Z =^{\diamondsuit} Y$. $\diamondsuit$-splitting is a special case of $\diamondsuit$-complementation modulo an ideal, which comes from the following definition in $\mathcal{E}$.

Definition 4.2. Let X and A be c.e. sets. X is *computable modulo* A if there is some c.e. Y such that $X \cap Y \subseteq A$ and $X \cup Y \cup A = \omega$. Equivalently, X is computable modulo A if there is some computable $R \subseteq X$ such that $X \subseteq A \cup R$.

The equivalence is straightforward, letting $Y = \overline{R}$ if the latter holds, and $R = X \setminus (Y \cup A)$ if the former holds. In $\mathcal{E}$, complementation and computability are equivalent, and hence it is clear that computablity mod A is $\{\subseteq\}$-definable in $\mathcal{E}$. In G, principal $\subsetneq$ complemented $\subsetneq$ computable, and complementation is the correct analogy to preserve $\{\subseteq\}$-definability.

Definition 4.3. Let $G = [0, M]$ and let $X, A \in G$. X is *$\diamondsuit$-complemented ($\diamondsuit$-c) modulo* A if there exists $Y \in G$ such that $X \cap Y \subseteq^{\diamondsuit} A$ and $X \vee Y \vee A =^{\diamondsuit} M$. Otherwise we say X is *$\diamondsuit$-noncomplemented ($\diamondsuit$-nc) modulo* A. When A is 0, we may just say X is $\diamondsuit$-c or $\diamondsuit$-nc.

We may also speak of $\diamondsuit$-(non)complementation within another ideal Z, where Z then takes the place of M above. A $\diamondsuit$-split of I is exactly an ideal that is $\diamondsuit$-c mod 0 in I, and we may use both terms. To say

X is $\diamondsuit$-c modulo A is Σ^0_3-definable, just as computability modulo A is, and $\{\subseteq\}$-definable in $I(Q)$.

Note that we may choose the same principal ideal to witness both the $\subseteq^\diamondsuit$ and the $=^\diamondsuit$ in Definition 4.3. Along the lines of the equivalence in Definition 4.2, if there is some $\diamondsuit$-c (modulo 0) c.e. ideal $R \subseteq X$ such that $X \subseteq^\diamondsuit A \vee R$, we can conclude X is $\diamondsuit$-c mod A. This is an equivalent characterization, ignoring the cosmetic circularity; again it is shown via letting Y be a $\diamondsuit$-complement of R to obtain the former from the latter, and $R = X \setminus (Y \vee A)$ to obtain the latter from the former, where this R is $\diamondsuit$-c via $(Y \vee A) \setminus X$.

Proposition 4.4. *There exists a $\diamondsuit$-nc ideal I within any $Z \neq^\diamondsuit 0$, modulo any $A \neq^\diamondsuit Z$, and of any specified Turing degree $\boldsymbol{d}$.*

Proof. Working in $2^{<\omega}$, let $\{z_i : i \in \omega\}$ be a disjoint generating set for Z. To make I of degree at least $\boldsymbol{d}$, let D be a c.e. ideal of that degree and f a computable isomorphism from $2^{<\omega}$ to $\langle z_0\rangle$; set $I \cap \langle z_0\rangle = f(D)$. If $\deg(Z) \leq \boldsymbol{d}$ we may put the cones off $z_i^\frown 0^\omega$ for all $i \geq 1$ (i.e., $\{\langle z_i^\frown 0^n 1\rangle : i \geq 1, n \in \omega\}$) into I. If not, use all z_i, but only the cones $\langle z_i^\frown 0^n 1\rangle$ such that $|z_i^\frown 0^n 1|$ is greater than the stage at which z_i is enumerated. This I is $\diamondsuit$-nc in Z modulo any $A \neq^\diamondsuit Z$: if $\hat{I}$ is a c.e. ideal such that $I \vee \hat{I} \vee A =^\diamondsuit Z$, then since $A \neq^\diamondsuit Z$, it can only cover coinfinitely-many of the paths $z_i^\frown 0^\omega$. Hence $\hat{I}$ must cover the remaining infinitely-many, meaning $I \cap \hat{I} \not\subseteq^\diamondsuit A$. □

When we refer to an ideal built "as in Proposition 4.4" with no reference to Z or $\boldsymbol{d}$, we mean $Z = M$ and $\boldsymbol{d} = \mathbf{0}$. Many of these ideals have a "noncomplementation preservation" property we will need for Lemma 7.8.

Lemma 4.5. *If I is constructed as in Proposition 4.4 with* $\deg(Z) \leq \boldsymbol{d}$, *then if $B \subseteq Z$ is $\diamondsuit$-nc mod A in Z, so is $B \cap I$.*

Proof. Suppose $B \cap I$ is $\diamondsuit$-c in Z modulo A via $\hat{B}$ and principal ideal P; without loss of generality let P contain $\langle z_0\rangle$. We claim that $\hat{B}$ is in fact a $\diamondsuit$-complement of B itself. The proof boils down to the fact that there is very little left over in Z once I is taken out.

Clearly since $(B\cap I)\vee\hat{B}\vee A\vee P = Z\vee P$, we have $B\vee\hat{B}\vee A\vee P = Z\vee P$. Suppose there is some $\sigma \in (B \cap \hat{B}) - (A \vee P)$. Necessarily it is $z_i^\frown 0^n$ for some i, n, since P contains $\langle z_0\rangle$ and $\deg(Z) \leq \boldsymbol{d}$. Therefore $\langle z_i^\frown 0^n\rangle \subseteq B \cap \hat{B}$, and for all k, $\langle z_i^\frown 0^{n+k} 1\rangle \subseteq I$. Therefore $\langle z_i^\frown 0^{n+k} 1\rangle$ witnesses $\hat{B}\cap(B\cap I) \not\subseteq A\vee P$, contradicting the choice of $\hat{B}$ and P. □

4.2. **$\diamondsuit$-splitting theorems.** We next prove that the standard splitting theorems, which preserve noncomplementation in sets, translate and extend to preserve $\diamondsuit$-noncomplementation in ideals. We will treat the Friedberg and Owings theorems separately because we will need the full strength of Owings in one setting, and the construction method of Friedberg in another. Both theorems are uniform and both still give genuine splits, not merely $\diamondsuit$-splits. Because of this, if the principal ideal P witnesses that B is $\diamondsuit$-c in I (modulo C for Owings), then P also witnesses the splits of B are $\diamondsuit$-c in I (mod C).

4.2.1. *The $\diamondsuit$-Friedberg Splitting Theorem.* The Friedberg Splitting Theorem (see Soare [12], X.2.1) states that a c.e. set splits into disjoint subsets such that if the original set was noncomplemented relative to (that is, inside) some c.e. W, so are the splits. The important aspect of the proof for Cholak and Harrington [6] is that the split X of Y is built such that for all e, if $W_e \searrow Y$ is infinite, $W_e \searrow X$ is also infinite. Achieving this accomplishes the full statement of Friedberg Splitting in $\mathcal{E}$ because if $W_e \cap Y$ is not complemented in W_e, then $W_e \searrow Y$ is infinite. Our translation of the proof builds X such that if $I_e \searrow Y \neq^{\diamondsuit} 0$, then $W_e \searrow X \neq^{\diamondsuit} 0$. The following lemma shows that is sufficient for the desired result in G.

Lemma 4.6. *Let $X, Y \in G$. If $X \cap Y$ is $\diamondsuit$-nc in X, then $X \searrow Y \neq^{\diamondsuit} 0$.*

Proof. Suppose that $X \searrow Y =^{\diamondsuit} 0$. We show $X \cap Y$ is $\diamondsuit$-c in X by the c.e. ideal $X \setminus Y$. Recalling $X \cap Y = (X \searrow Y) \vee (Y \searrow X)$, we see $X \cap Y =^{\diamondsuit} Y \searrow X$. Hence $(X \cap Y) \cap (X \setminus Y) =^{\diamondsuit} (Y \searrow X) \cap (X \setminus Y)$, which is clearly 0. We also have $(X \setminus Y) \vee (X \cap Y) = X$, so $X \setminus Y$ is a $\diamondsuit$-complement of $X \cap Y$ in X. □

The same proof actually shows that if $X \cap Y$ is $\diamondsuit$-nc mod A in X, then $X \searrow Y \not\subseteq^{\diamondsuit} A$.

Theorem 4.7 ($\diamondsuit$-Friedberg Splitting)**.** *For any $\diamondsuit$-nc $B \in G$, there are $\diamondsuit$-nc ideals A_0, A_1 such that $A_0 \cap A_1 = 0$ and $A_0 \vee A_1 = B$. Moreover, if $I \in G$ is such that $B \cap I$ is $\diamondsuit$-nc in I, then $A_i \cap I$ is $\diamondsuit$-nc in I for $i = 0, 1$.*

Proof. Let $\{b_0, b_1, \ldots\}$ be a disjoint generating set for B. We construct A_0, A_1 to be a split of B by enumerating each b_s into exactly one A_i, at stage s. Noting the final condition includes that A_i is $\diamondsuit$-nc in M, we meet it via the following requirements for all $e \in \omega$, $m \in M$, and $i = 0, 1$:

$$R_{\langle e,i,m\rangle} : I_e \searrow B \neq^{\diamondsuit} 0 \;\Rightarrow\; I_e \cap A_i \not\subseteq \langle m\rangle.$$

By Lemma 4.6, for any e such that $I_e \cap B$ is $\diamondsuit$-nc in I_e, the antecedent will hold. We will use the consequent to show $I_e \cap A_i$ is also $\diamondsuit$-nc in I_e.

Construction.

Stage $s = 0$: Enumerate b_0 into A_0.

Stage $s+1$: Choose the least $\langle e, i, m\rangle$ such that $\langle b_{s+1}\rangle \cap I_{e,s} \not\subseteq \langle m\rangle$ but $A_{i,s} \cap I_{e,s} \subseteq \langle m\rangle$ and enumerate b_{s+1} into A_i. [Notice that some $x \in \langle b_{s+1}\rangle$ is in $I_e \searrow B$ outside $\langle m\rangle$.] If there is no such triple enumerate b_{s+1} into A_0.

Verification.

We first show all requirements are met. It is clear that each requirement acts at most once. Suppose toward a contradiction that $\langle e, i, m\rangle$ is the least triple such that $R_{\langle e,i,m\rangle}$ is not met, and let t be a stage such that all earlier requirements have ceased acting. By assumption on $R_{\langle e,i,m\rangle}$, $I_e \searrow B \neq^{\diamondsuit} 0$ but for all $s \geq t$ $\langle b_{s+1}\rangle \cap I_{e,s} \subseteq \langle m\rangle$. This asserts $I_e \searrow B \subseteq (I_e \searrow B)_t \vee \langle m\rangle$, which is clearly a contradiction.

Now, suppose I is a c.e. ideal such that $B \cap I$ is $\diamondsuit$-nc in I but $A_i \cap I$ is $\diamondsuit$-c in I by X, witnessed by m. We must have $(I \cap X) \cap B$ $\diamondsuit$-nc in $I \cap X$; any $\diamondsuit$-complement would also be a $\diamondsuit$-complement of $I \cap B$ in I. Hence by Lemma 4.6, $(I \cap X) \searrow B \neq^{\diamondsuit} 0$, so the construction gives $I \cap X \cap A_i \not\subseteq \langle m\rangle$, contradicting the assumptions on m and X. □

4.2.2. *$\diamondsuit$-Friedberg Splitting modulo A.* The following lemma translates a strengthened version of Lemma 6.3 in [6]. It is an extension to the $\diamondsuit$-Friedberg Splitting Theorem and will be needed for Theorem 7.2. We cannot use the Friedberg proof technique to work modulo some set A, but if we build A concurrently and restrain elements in $I_e \cap X$ from entering A, to meet the requirements below, we can guarantee X is $\diamondsuit$-nc modulo A.

Lemma 4.8. *Assume Y is $\diamondsuit$-nc mod A. If we build $X \subseteq Y$ to meet the requirements*

$$R_{e,k}: \text{ if } I_e \searrow Y \neq^{\diamondsuit} 0 \text{ then } I_e \cap X \not\subseteq A \vee \langle m_k\rangle,$$

then X is $\diamondsuit$-nc mod A.

Proof. If $I_e \vee Y \vee A \vee \langle m_k\rangle \neq M$, then regardless of how we build X, it will not be $\diamondsuit$-c mod A via I_e and $\langle m_k\rangle$. Assume, then, that $I_e \vee Y \vee A \vee \langle m_k\rangle = M$. By assumption on Y, $I_e - (A \vee Y)$ must be a nonprincipal collection of elements. If $I_e \searrow Y \subseteq \langle m_j\rangle$ for some j, then $(I_e \setminus Y)$ is a $\diamondsuit$-complement mod A to Y, witnessed by $\langle m_k, m_j\rangle$, which is a contradiction. Hence $I_e \searrow Y \neq^{\diamondsuit} 0$ and we make $I_e \cap X \not\subseteq A \vee \langle m_k\rangle$, showing again that X is not $\diamondsuit$-c mod A via I_e and $\langle m_k\rangle$. □

4.2.3. *The $\diamondsuit$-Owings Splitting Theorem.* The Owings Splitting Theorem states that a c.e. set that is noncomplemented in an interval may be split into two disjoint c.e. sets that are noncomplemented in the same interval (see [12] X.2.5). The translation of Owings Splitting to $G^\diamondsuit$ holds as a corollary of the isomorphism between $G^\diamondsuit$ and $\mathcal{E}^*$, and the direct translation to G holds as a corollary to that [13]. The $\diamondsuit$-complementation version is as follows.

Theorem 4.9 ($\diamondsuit$-Owings Splitting)**.** *Suppose $C \subseteq B$ are elements of G such that B is $\diamondsuit$-nc modulo C. Then there exist c.e. ideals $A_0, A_1 \subseteq M$ such that*
1. $A_0 \cap A_1 = 0$;
2. $A_0 \vee A_1 = B$;
3. $A_i \vee C$ *is $\diamondsuit$-nc modulo C, $i = 0, 1$;*
4. *For any c.e. ideal $I \subseteq M$, if $B \subseteq I$ and relative to I, B is $\diamondsuit$-nc modulo C, then likewise $A_i \vee C$ is $\diamondsuit$-nc modulo C relative to I for $i = 0, 1$.*

Proof. Let $\{b_0, b_1, \ldots\}$ be a disjoint generating set for B. To achieve (1) and (2) we will enumerate each b_s into exactly one of A_0, A_1, at stage s. For the rest, for each $e \in \omega$, $m \in M$, and $i = 0, 1$ we have the requirement

$$R_{\langle e,i,m\rangle} : I_e \vee A_i \vee C \vee \langle m\rangle = M \;\Rightarrow\; I_e \cap A_i \not\subseteq C \vee \langle m\rangle.$$

Though this appears only to address property (3), ensuring I_e is not a $\diamondsuit$-complement of A_i mod C, we will use a strategy that meets (4) as well. For each $\langle e, i, m\rangle$ we have a function $g(e, i, m, s)$.

Construction.

Stage 0: Enumerate b_0 into A_0 and set $g(e, i, m, 0) = 0$ for all e, i, m.
Stage $s + 1$:

1. Perform this step for every triple $\langle e, i, m\rangle \leq s$. If there is an $x \in P_{\leq g(e,i,m,s)}$ such that $x \in I_{e,s} \cap A_{i,s} \cap \overline{\langle m\rangle} \cap \overline{C_s}$ (the last ideal is well-defined since C_s is principal at every stage s), set $g(e, i, m, s + 1) = g(e, i, m, s)$. Otherwise set $g(e, i, m, s + 1) = s + 1$.
2. Choose the least triple $\langle e, i, m\rangle$ such that $\langle b_{s+1}\rangle \cap I_{e,s} \not\subseteq C_s \vee \langle m\rangle$, $A_{i,s} \cap I_{e,s} \subseteq C_s \vee \langle m\rangle$, and $b_{s+1} \trianglelefteq m_{g(e,i,m,s)}$, and enumerate b_{s+1} into A_i. If there is no such triple enumerate b_{s+1} into A_0.

Verification. It is clear from the construction that (1) and (2) hold. We will show (4) holds, which gives (3) as a special case.

Suppose for some e, i, m, and ideal I, A_i is $\diamondsuit$-c in $[C, I]$ by I_e, witnessed by m. We must show B is $\diamondsuit$-complemented in $[C, I]$.

Choose s' large enough that for all $\langle e', i', m'\rangle < \langle e, i, m\rangle$ such that $\lim_s g(e', i', m', s) < \infty$, $\lim_s g(e', i', m', s) = g(e', i', m', s')$. Let z be the maximum of those limit values. Choose $s'' \geq s'$ such that $b_s \rhd m_z$ for all $s \geq s''$. Define the c.e. ideal

$$V_e = \bigvee_{s \geq s''} (I_{e,s} \cap \overline{B_s} \cap P_{\leq g(e,i,m,s)}).$$

Since $\lim_s g(e, i, m, s) = \infty$ by the assumption that $I_e \cap A_i \subseteq C \vee \langle m\rangle$, $V_e \vee B = I_e \vee B$ and hence $V_e \vee B =^\diamondsuit I \vee B$ because $A_i, C \subseteq B$.

We now show $V_e \cap B \subseteq^\diamondsuit C$, witnessed by $P_{\leq s''}$. Suppose for a contradiction that $x \in V_e \cap B$, $x \notin C$; say $x \in I_{e,s} \cap \overline{B_s} \cap P_{\leq g(e,i,m,s)}$ and $x \in \langle b_{s+1}\rangle$ for some $s \geq s''$. Then by the construction, at stage $s+1$, b_{s+1} must be chosen by $R_{\langle e',i',m'\rangle}$ for some $\langle e', i', m'\rangle \leq \langle e, i, m\rangle$. If $\langle e', i', m'\rangle = \langle e, i, m\rangle$, then x enters A_i and hence must be in $A_i \cap I_e$ which is a subideal of C by assumption on I_e. If $\langle e', i', m'\rangle < \langle e, i, m\rangle$, then by assumption on s'' we must have $\lim_s g(e', i', m', s) = \infty$, so x is not a permanent witness for $R_{\langle e',i',m'\rangle}$ and hence must later enter C. Therefore $V_e \vee C$ demonstrates B's complementation in $[C, I]$. □

5. Patterns, players, and realization

The definitions below of the patterns $\mathcal{P}_i$ and the formula $\varphi_\mathcal{P}$ are essentially identical to the definitions made in Cholak and Harrington [6], though drastically condensed and with the necessary changes made to translate them from sets to ideals.

For us, a *pattern* $\mathcal{P} = (\mathcal{T}, \mathcal{R}, \mathcal{B}, \ell)$ is a finite tree $\mathcal{T}$ with two distinguished (finite) sets $\mathcal{R}, \mathcal{B}$ of nodes, and a function ℓ from $\{0, 1, 2, 3\}$ to the power set of $\mathcal{T}$. There is an associated array of ideals $\vec{U} = \{U_0, U_1, U_2, U_3\}$. We have distinguished patterns $\mathcal{P}_i$, $i \in \omega$, with the following nodes and associated ideals.

Node		Children (L, R)		Ideal
b_0		b_2^0, b_1		
b_1				U_0
b_2^k	$(0 \leq k \leq i)$	r_2^k, r_1^k		
r_1^k	$(0 \leq k \leq i)$			U_1
r_2^k	$(0 \leq k \leq i)$	b_2^{k+1}, b_3^k	$(b_2^{i+1} = b_4)$	
b_3^k	$(0 \leq k \leq i)$			U_2
b_4		b_5 (L)		
b_5				U_3

For $\mathcal{P}_i$, the set $\ell(j)$ contains all nodes associated to U_j. The set $\mathcal{R}$ contains all b_2^k for $0 \leq k \leq i$ ($\mathcal{R}$ excludes b_4). The set $\mathcal{B}$ contains all b-nodes, including b_2^k.

For every $\mathcal{P}_i$ the root of the tree is b_0. We use $d(q)$ to denote the set of children of node q and $u(q)$ to denote the parent of q. $\mathcal{T}$ is partially ordered by height: $q < u(q)$, and if $p \in d(q)$, $p < q$.

These patterns $\mathcal{P}_i$ are examples of *special $\mathcal{L}$-patterns* ([6] Definition 3.12; $\mathcal{L}$ is the set indexing $\vec{U}$, which is called an *$\mathcal{L}$-interpretation*). Many of the results are stated for special $\mathcal{L}$-patterns in general because they do not rely on particular properties of the $\mathcal{P}_i$.

The main proof will consist of forcing or forbidding certain properties to hold of arrays of ideals associated to the patterns, as follows. We think of elements of M as balls.

Definition 5.1. Given pattern $\mathcal{P}$, an $\mathcal{L}$-interpretation $\vec{U}$, an array $\vec{B} = \langle B_p : p \in \mathcal{B}\rangle$, and c.e. ideals A and C, the formula $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}, C)$ says there exists an array $\vec{D} = \langle D_p : p \in \mathcal{T}\rangle$ such that all of the following hold.

(i) We work in C: $(\forall p \in \mathcal{T})[D_p \subseteq C]$.
(ii) $\vec{D}$ represents balls flowing down the tree, modulo the contents of A: $(\forall p \leq q \in \mathcal{T})[D_p \subseteq D_q \vee A]$.
(iii) Every ball that reaches a node in $\mathcal{R}$ continues down the tree, modulo A: $(\forall q \in \mathcal{R})\,[D_q \vee A = \bigvee\{D_p : p \in d(q)\} \vee A]$.
(iv) The array $\vec{B}$ determines movement into $\mathcal{B}$-nodes, modulo A: $(\forall p \in \mathcal{B})\,\big[D_p \vee A = (B_p \cap D_{u(p)}) \vee A\big]$.
(v) Every ball that reaches a node in $\ell(j)$ is in U_j, modulo A: $(\forall j < 4)(\forall p \in \ell(j))\,[D_p \subseteq U_j \vee A]$.
(vi) Item (v) accounts for all balls in both $\vec{D}$ and $\vec{U}$, modulo A: $(\forall j < 4)(\forall q \in \mathcal{T})\,[(D_q \cap U_j) \vee A = \bigvee\{D_p : p \leq q \,\&\, p \in \ell(j)\} \vee A]$.

Satisfaction of $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}, C)$ may be viewed as a game. Player BLUE creates $\vec{B}$, called a *$\mathcal{B}$-interpretation* or a *BLUE strategy*, and player RED reacts with $\vec{D}$, called an *interpretation of $\mathcal{P}$ over $\vec{U}$*. RED must respect both $\vec{B}$ and $\vec{U}$, but has some control at $\mathcal{R}$ nodes: item (iii) above says all balls must move down from such nodes, but their children are not in $\mathcal{B}$, so BLUE cannot force the direction of movement.

Note for a fixed pattern $\mathcal{P}$ all $\vec{D}$ and $\vec{B}$ are indexed so we may quantify over them. Just as in [6], $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}, C)$ is an arithmetically Σ^0_3 formula (recall $\mathcal{R}, \mathcal{B}$ are finite sets) which may also be written in the language $\{\subseteq\}$. In fact, it is an $\mathcal{L}(A)$ property, where here $\mathcal{L}(A) = \{A \vee B : B \in G\}$. It is easy to check that the conditions above hold for A, $\vec{U}$, $\vec{B}$, and C if and only if for an isomorphism Φ between $\mathcal{L}(A)$ and $\mathcal{L}(F)$, they also hold of F, $\Phi(\vec{U})$, $\Phi(\vec{B})$, and $\Phi(C)$.

What we will actually force or forbid is more than just $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}, C)$.

Definition 5.2. *Realization* of the pattern $\mathcal{P}$ by $\vec{U}$, A, W means for all possible BLUE strategies $\vec{B}$ there is a $\diamondsuit$-split C of W such that $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}, C)$ holds and C is $\diamondsuit$-nc modulo A.

Realization of a *fixed* pattern $\mathcal{P}$ is uniformly definable in $\{\subseteq\}$ and arithmetically Π^0_5, since $\varphi_{\mathcal{P}}$ and $C \sqsubseteq^{\diamondsuit} W$ are Σ^0_3 and C $\diamondsuit$-nc mod A is Π^0_3.

Theorem 6.1 below, translating [6] Theorem 4.1, eliminates the need for the universal quantifier on $\vec{B}$ by establishing a "universal" BLUE strategy for any given pattern $\mathcal{P}$ and collection of possible realizers A, W, $\vec{U}$. In the construction for Theorem 7.2, there will be one pattern in which we play for RED and build C and $\vec{D}$ in reaction to that BLUE strategy to force realization. In the remaining patterns, we will play for BLUE, trying to defeat arrays $\vec{D}$ presented by RED or show that the C at hand is either not a $\diamondsuit$-split of W or is $\diamondsuit$-c mod A.

We will use the following definition, identical to 6.6 in [6].

Definition 5.3. Fix a c.e. ideal A and $\mathcal{L}$-interpretation $\vec{U}$ for $\mathcal{L} = \{0, 1, 2, 3\}$. Let $\mathcal{J}_{A,\vec{U}}$ be the set of special $\mathcal{L}$-patterns $\mathcal{P}$ such that $\exists Y(\vec{U}, A, Y$ realize $\mathcal{P})$.

For any fixed $\mathcal{P}$, the statement "$\mathcal{P} \in \mathcal{J}_{A,\vec{U}}$" is $\{\subseteq\}$-definable in $I(Q)$ and hence invariant in $I(Q)$ and G. In fact, it is again an $\mathcal{L}(A)$ property. Additionally, it has only an extra existential quantifier in front of the definition for realization. The universal BLUE strategy of Theorem 6.1 will show realization is in fact only Σ^0_4, and using Theorem 6.10 or highness, we may reduce the complexity yet again to Σ^A_3. This means "$\mathcal{P} \in \mathcal{J}_{A,\vec{U}}$", though a priori Σ^0_6, is also Σ^A_3.

Finally, we collect some basic facts about $\varphi_{\mathcal{P}}$ and realization for ease of reference.

Lemma 5.4. (i) *For $A \subseteq \hat{A}$, $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}, C) \Rightarrow \varphi_{\mathcal{P}}(\hat{A}, \vec{U}, \vec{B}, C)$.*
(ii) *For $F \subseteq C$, $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}, C) \Rightarrow \varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}, F)$.*
(iii) *$[(\forall k < 4)(V_k \cap W = U_k \cap W)] \Rightarrow$*
$$[\vec{U}, A, W \text{ realize } \mathcal{P} \Leftrightarrow \vec{V}, A, W \text{ realize } \mathcal{P}].$$

6. Reducing arithmetic complexity

6.1. **A universal BLUE strategy.** In this section we reduce the complexity of realization from Π^0_5 to Σ^0_4, and hence "$\mathcal{P} \in \mathcal{J}_{A,\vec{U}}$" from Σ^0_6 to Σ^0_4.

Theorem 6.1. *Fix $\mathcal{L}, \vec{U}, \mathcal{P}, A, W$. Uniformly in $\vec{U}$, $\mathcal{P}$, A, W (indexed by e) there exists a $\mathcal{B}$-interpretation $\vec{B}_e$ such that the following are equivalent:*

(1) *There exists a $\diamondsuit$-split S of W such that S is $\diamondsuit$-nc modulo A and $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}_e, S)$.*
(2) *$\vec{U}, A, W$ realize $\mathcal{P}$.*

Proof. It is clear that (2) implies (1), so assume (1). We will show the Recursion Theorem allows us to assume we have an index e for $\vec{B}_e$.

By (1), we know there exists a $\diamondsuit$-split S of W which is $\diamondsuit$-nc modulo A and such that $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}_e, S)$. We first use a tree of strategies to create manageable subsplits of all $\diamondsuit$-splits of W; the tree is $2^{<\omega}$. We may enumerate a list of all quadruples $(\tilde{S}_i, \hat{S}_i, z_i, \vec{D}_i)$, where $\tilde{S}_i, \hat{S}_i \in G$, $z_i \in M$, and $\vec{D}_i$ is an appropriately sized array of c.e. ideals, and assign all nodes of length i to the i^{th} quadruple. We will ensure for such a node α that if α is on the true path f, then $\alpha^\frown 0 \subset f$ if and only if $\hat{S}_i$ and z_i witness $\tilde{S}_i$ is a $\diamondsuit$-split of W and $\vec{D}_i$ witnesses $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}_e, \tilde{S}_i)$ (we do not yet worry about $\diamondsuit$-complementation modulo A). This is arithmetically Π^0_2 information, and hence is equivalent to some $\forall x \exists s \Theta(x, s)$ where Θ is computable and may be found uniformly from the given parameters. We use Θ for a sort of length of agreement function:

$$\ell_\alpha(s) = \max\{x : (\forall y < x)(\exists s_y < s)\Theta(y, s_y)\}.$$

At each stage s, define the true path approximation f_s inductively, restricting to $|f_s| \leq s$. If $\alpha \subseteq f_s$, let t be the latest stage $< s$ such that $\alpha^\frown 0 \subseteq f_t$ (or 0 if there is no such stage), and let $\alpha^\frown 0 \subseteq f_s$ iff $\ell_\alpha(s) > \ell_\alpha(t)$ (that is, if s is α-expansionary); otherwise $\alpha^\frown 1 \subseteq f_s$.

The node $\beta = \alpha^\frown 0$ effectively builds $S_\beta \subseteq \tilde{S}_i \cap W$. It uses two numbers d_β, $u_\beta(s)$ to define an "interval" from which it may put elements into S_β. At the first stage (after initialization) that $\beta \subseteq f_s$, set d_β to be a large unused number and $u_\beta(s)$ to be even larger; halt the stage. If $\beta \subseteq f_t$ for $t > s$ and both values are defined, set $u_\beta(t)$ to a new larger value, and otherwise let $u_\beta(t) = u_\beta(t-1)$. The value of d_β will stabilize and u_β limit to infinity exactly if β is on the true path. As usual, nodes are initialized if the approximation to the true path ever passes to their left or terminates above them.

For every β with a defined interval $(d_\beta, u_\beta]$, where β is assigned to $(\tilde{S}_i, \hat{S}_i, z_i, \vec{D}_i)$ and so is building $S_\beta \subseteq \tilde{S}_i \cap W$, at stage s put all x satisfying the following four conditions into S_β:

(i) $x \in P_{\leq u_\beta(s)} - P_{\leq d_\beta}$
(ii) $x \in \tilde{S}_{i,s} \cap W_s$

(iii) $(\forall\gamma)(\langle x\rangle \cap S_{\gamma,s} = 0)$
(iv) $(\forall\gamma^\frown 0 \subset \beta)(x \in (\hat{S}_{|\gamma|,s} - \langle z_{|\gamma|}\rangle))$

The conditions say x is in β's interval and in the ideal it is emulating, won't cause intersection with another split and won't cause theft from a higher priority split. Condition (iv) does not blockade us because we need only at most one node related to a given $\tilde{S}_i$ to construct S_β. This completes the tree construction.

From the enumeration $\{\vec{B}_j\}$ of all $\mathcal{B}$-interpretations and the S_β built on the tree, we define $\vec{B}_e$. Using the $\diamondsuit$-Owings Splitting Theorem, split each S_β, $\beta \in 2^{<\omega}$, effectively into $S_{\beta,j}$, $j \in \omega$, preserving $\diamondsuit$-noncomplementation modulo A if S_β has that property. Define $B_{e,p} \cap S_{\beta,j} = B_{j,p} \cap S_{\beta,j}$, where $B_{e,p}$ is the p^{th} ideal of $\vec{B}_e$, and likewise for $B_{j,p}$. By the definition of φ, since $S_{\beta,j} \subseteq S_\beta$ we have

$$\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}_e, S_\beta) \;\Rightarrow\; \varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}_e, S_{\beta,j})$$

for all j, and by the definition of $\vec{B}_e$,

$$\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}_e, S_{\beta,j}) \;\Rightarrow\; \varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}_j, S_{\beta,j}).$$

Since the construction of S_β is uniformly effective in β and the S_β's are constructed to be pairwise disjoint, the construction of $\vec{B}_e$ is uniform in $\vec{U}$, $\mathcal{P}$, A, and W, and hence the Recursion Theorem gives us the index e for $\vec{B}_e$.

All that remains is to show there is a $\diamondsuit$-noncomplemented $S_\beta \sqsubseteq^\diamondsuit W$ such that $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}_e, S_\beta)$ holds. Assuming (1), we know there is a quadruple $(\tilde{S}_i, \hat{S}_i, z_i, \vec{D}_i)$ such that $\tilde{S}_i$ is $\diamondsuit$-nc modulo A, $\hat{S}_i$ and z_i witness $\tilde{S}_i$ is a $\diamondsuit$-split of W, and $\vec{D}_i$ witnesses $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}_e, \tilde{S}_i)$; choose the least such. Let α be the length-i node on f and $\beta = \alpha^\frown 0 \subset f$. Since $S_\beta \subseteq \tilde{S}_i$, by Lemma 5.4 we have $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}_e, S_\beta)$ for this β, so we need $\diamondsuit$-noncomplementation and $\diamondsuit$-splitting. To this end, we define some auxiliary ideals.

For $\gamma^\frown 0 \subset \beta$ we know $\tilde{S}_{|\gamma|}$ must be $\diamondsuit$-c modulo A, since $\ell_\gamma \to \infty$ gives $\hat{S}_{|\gamma|}$, $z_{|\gamma|}$ witness $\tilde{S}_{|\gamma|} \sqsubseteq^\diamondsuit W$ and $\vec{D}_{|\gamma|}$ witnesses $\mathcal{P}(A, \vec{U}, \vec{B}_e, \tilde{S}_{|\gamma|})$. Hence $\bigvee\{\tilde{S}_{|\gamma|} : \gamma^\frown 0 \subset \beta\}$ is a $\diamondsuit$-split of W that is $\diamondsuit$-c modulo A, and $\tilde{S} := \tilde{S}_i \cap \bigcap\{\hat{S}_{|\gamma|} : \gamma^\frown 0 \subset \beta\}$ is a $\diamondsuit$-split of W that is $\diamondsuit$-nc modulo A (since $\tilde{S}_i$ is $\diamondsuit$-nc mod A).

Define a pair of c.e. ideals Q, R as follows. Assume t is a stage after which β is never initialized again. Given $x \notin P_{\leq d_\beta}$, wait for a stage $s \geq t$ such that $x \in P_{\leq u_\beta(s)} - P_{\leq u_\beta(s-1)}$, or if $s = t$, $x \in P_{\leq u_\beta(t)} - P_{\leq d_\beta}$. If there is any node γ such that $x \in S_{\gamma,s}$ (β and its predecessors are prohibited from using x), put x into R. If $\langle x\rangle \cap S_{\gamma,s} = 0$ for all γ, put

x into Q. Note that while not every element of $P_{\leq u_\beta(s)} - P_{\leq u_\beta(s-1)}$ will be parceled out, a generating set will be. R and Q are disjoint, and $R \vee Q =^\diamondsuit M$, witnessed by $P_{\leq d_\beta}$.

By construction, $S_\beta =^\diamondsuit \tilde{S} \cap Q$; this gives $S_\beta \sqsubseteq^\diamondsuit W$. Suppose Y is a $\diamondsuit$-complement of S_β modulo A. We show $\tilde{S}$ is also $\diamondsuit$-c modulo A, by $Y \cap (\hat{S} \vee Q)$. First consider the join:

$$[Y \cap (\hat{S} \vee Q)] \vee \tilde{S} \vee A = (Y \vee \tilde{S} \vee A) \cap (\hat{S} \vee \tilde{S} \vee Q \vee A).$$

The first ideal in the intersection is $=^\diamondsuit M$ because $\tilde{S}$ contains S_β. The second is $=^\diamondsuit M$ because $\hat{S} \vee \tilde{S}$ is W and hence contains R. Next the meet:

$$Y \cap (\hat{S} \vee Q) \cap \tilde{S} = (Y \cap \hat{S} \cap \tilde{S}) \vee (Y \cap Q \cap \tilde{S}).$$

The first ideal in the join is $=^\diamondsuit 0$ since $\hat{S}$ and $\tilde{S}$ $\diamondsuit$-split W. The second is $\subseteq^\diamondsuit A$ by assumption on Y, because $Q \cap \tilde{S} =^\diamondsuit S_\beta$. Hence by contradiction, S_β is $\diamondsuit$-nc modulo A, and we have realization of $\mathcal{P}$ by $\vec{U}$, A, W. $\square$

Corollary 6.2. *Whether $\vec{U}, A, W$ realize $\mathcal{P}$ is Σ^0_4.*

6.2. **True stage enumerations.** In this section we reduce the complexity of realization, and hence of "$\mathcal{P} \in \mathcal{J}_{A,\vec{U}}$", from Σ^0_4 to Σ^A_3.

Definition 6.3. Recall from §2.1 that we fix an enumeration of M and use it to define the ordering $x \lhd y$ for $x, y \in M$. Let $\{a_0 \lhd a_1 \lhd \dots\}$ be the induced ordering of an ideal A's set-theoretic complement $M - A$, and for an enumeration $\{A_s\}_{s\in\omega}$ of A, label $M - A_s$ as $\{a^s_0 \lhd a^s_1 \lhd \dots\}$. $\{A_s\}_{s\in\omega}$ is a *true stage enumeration* if for infinitely-many s, $a^s_s = a_s$.

The following lemma holds by the same proof as Lemma 7.2 in [6], as it does not depend on the algebraic properties of sets versus ideals.

Lemma 6.4. *Any ideal A that is not high has a true stage enumeration.*

Note that being a true stage for A and some fixed $\{A_s\}$ is Π^0_1 and Δ^A_0. The following theorem translates [6] Theorem 7.3. Recall Definition 4.1 of $\exists^{np} x\varphi(x)$, that there exists a collection of x satisfying $\varphi(x)$ that is not contained in any principal ideal. In particular, $\exists^{np}$ is Π^0_2.

Theorem 6.5. *Suppose A has a true stage enumeration $\{A_s\}$. Uniformly in e there exists a c.e. ideal $F_e \subseteq I_e$ with a computable enumeration $\{F_{e,s}\}$ such that for all $\diamondsuit$-splits S of I_e,*

(1) if S is $\diamondsuit$-nc modulo A, then $(\star)$;
(2) if $(\star)$, then $S \cap F_e$ is $\diamondsuit$-nc modulo A;

where $(\star)$ *is the statement*

$$(\exists^{np} x = m_t)(\exists s > t)[s \text{ is a true stage} \,\&\, x \notin A_s \\ \&\, x \in S \,\&\, \langle x \rangle \cap F_{e,s} = 0].$$

Proof. We drop the index e and assume by the Recursion Theorem that we have an index for F.

The idea behind meeting (1) is that if $(\star)$ fails for S, outside of some $P_{\leq s^*}$, if $s > s^*$ is a true stage such that $x \lhd m_s$, $x \in S$, and $\langle x \rangle \cap F_s = 0$ (in particular $x \notin F$), then $x \in A_s$ and hence $x \in A$. Therefore $\neg(\star) \Rightarrow S \subseteq^{\diamondsuit} A \vee F$. If we keep F (or rather, $F \cap S$) small, we get $S \subseteq^{\diamondsuit} A$ and hence S $\diamondsuit$-c modulo A. For (2), we try to force $S \cap F$ to be $\diamondsuit$-nc mod A, and show that if we cannot, $(\star)$ must fail.

To meet (1), let $\{\tilde{S}_i, \hat{S}_i, y_i\}$ be an enumeration of $G \times G \times M$. Whether $\hat{S}_i$, y_i witness $\tilde{S}_i \sqsubseteq^{\diamondsuit} I$ is Π^0_2. Since $(\star)$ is Π^0_3, $\neg(\star)$ is Σ^0_3 and we may assume it is written in the form $(\exists j)\Theta(i,j)$ for some $\Theta \in \Pi^0_2$. We have the requirements

$$N_{i,j} : [\hat{S}_i, y_i \text{ witness } \tilde{S}_i \sqsubseteq^{\diamondsuit} I \,\&\, \Theta(i,j)] \implies \\ (\exists R \subseteq F)[R \ \diamondsuit\text{-c modulo } 0 \,\&\, F \subseteq^{\diamondsuit} \hat{S}_i \vee A \vee R].$$

We will use R, which will come from the tree construction, to show $\tilde{S}_i$ is $\diamondsuit$-c mod A. To meet $N_{i,j}$ we simply restrain enumeration of elements of $\tilde{S}_i$ into F as much as possible.

To meet (2), let $\{\tilde{R}_j, \hat{R}_j, z_j\}$ be an enumeration of $G \times G \times M$. It is Π^0_2 to tell whether $\hat{R}_j$ is a $\diamondsuit$-complement of $\tilde{R}_j$ witnessed by z_j. We have the requirements

$$P_{i,j,k} : [\hat{S}_i, y_i \text{ witness } \tilde{S}_i \sqsubseteq^{\diamondsuit} I \,\&\, \hat{R}_j, z_j \text{ witness } \tilde{R}_j \ \diamondsuit\text{-c modulo } 0] \implies \\ [\tilde{R}_j \not\subseteq (\tilde{S}_i \cap F) \vee \langle m_k \rangle \text{ or } \tilde{S}_i \cap F \not\subseteq \tilde{R}_j \vee A \vee \langle m_k \rangle].$$

We meet $P_{i,j,k}$ by searching for a ball $x \notin \langle z_j, m_k \rangle$ and a true stage s such that $x \unlhd m_s$, $x \in \tilde{S}_{i,s}$, $\langle x \rangle \cap F_s = 0$, and $x \notin A_s$; if $(\star)$ holds there must be a nonprincipal collection of such x. Once we have x, we feed bits of $\langle x \rangle$ into F as they enter $\hat{R}_j$, to obtain at least one disjunct of the consequent of P. Requirements with the same i and j may have comparable witnesses; all others will have disjoint witnesses.

The construction tree is $2^{<\omega}$; at each stage s there will be a finite true path approximation f_s of length no more than s. Extensions of the approximation will depend on whether a stage is expansionary; that is, on whether a particular length function has increased since the last time the node was on the approximation. To simplify some definitions below, let Ssplit(i,s) be the value

$$\max\left\{y : (\forall x < y)\left[m_x \in I_s \Rightarrow m_x \in \tilde{S}_{i,s} \vee \hat{S}_{i,s} \vee \langle y_i\rangle\right]\right\}$$

if $\tilde{S}_{i,s} \cap \hat{S}_{i,s} \subseteq \langle y_i\rangle$; otherwise let Ssplit$(i,s) = 0$. Define Rsplit$(j,s)$ likewise with $\tilde{R}_j, \hat{R}_j, z_j$.

All length-$2\langle i,j,k\rangle$ nodes δ are assigned to $P_{i,j,k}$. Define the function

$$\ell_\delta(s) = \max\{y : \text{Ssplit}(i,s) \geq y \ \&\ \text{Rsplit}(j,s) \geq y\}.$$

If $\delta \subseteq f_s$, $\delta^\frown 0 \subseteq f_s$ if s is δ-expansionary and $\delta^\frown 1 \subseteq f_s$ otherwise. The 0-children of P nodes will build F.

All length-$(2\langle i,j\rangle + 1)$ nodes α are assigned to $N_{i,j}$. Assume that since $\Theta(i,j)$ is Π^0_2 it is written in the form $\forall x \exists s \Theta^*(i,j,x,s)$ where Θ^* is Δ^0_0. Note that Θ and Θ^* may be found uniformly from i as required to apply the Recursion Theorem. Define the function

$$\ell_\alpha(s) = \max\{y : \text{Ssplit}(i,s) \geq y \ \&\ (\forall x < y)(\exists t \leq s)\Theta^*(i,j,x,t)\}.$$

If $\alpha \subseteq f_s$, then $\alpha^\frown 0 \subseteq f_s$ if s is α-expansionary and $\alpha^\frown 1 \subseteq f_s$ otherwise. The 0-children of N nodes will set restraint on enumeration into F.

The true path f is clearly the lim inf of the f_s. As usual, nodes are initialized whenever f_s passes to their left or terminates above them.

To build F, let $\delta \in 2^{<\omega}$ be of length $n = 2\langle i,j,k\rangle$ and let $\beta = \delta^\frown 0$. Let the current stage be s; assume $\beta \subseteq f_t$ for some $t \leq s$ and β has not been initialized since t. If β has no witness x_β to $P_{i,j,k}$ at s, look for $x = m_p$ meeting all of the following criteria:

$$x \notin \langle z_j,\ m_k\rangle,\ x \in \tilde{S}_{i,s},\ \langle x\rangle \cap F_s = 0,\ x \notin A_s,$$

$$\langle x\rangle \cap \langle x_\gamma\rangle = 0 \text{ for all } \gamma \subset \beta \ \&\ \gamma <_L \beta \text{ except } |\gamma| = 2\langle i,j,k'\rangle + 1,$$

$$x \in \hat{S}_{i',s} \text{ for all } i' \text{ s.t. } (\exists\gamma)[\gamma^\frown 0 \subseteq \beta \ \&\ |\gamma| = 2\langle i',j'\rangle + 1], \text{ and}$$

$$(\forall t')[p \leq t' \leq s \ \Rightarrow\ f_{t'} \not<_L \beta].$$

If there is such an x let the $\lhd$-least be x_β. If we ever see x_β enter A, it is released and β needs a new witness. If x_β is defined and we see some $y \in \langle x_\beta\rangle$ enter $\hat{R}_j$ at s, then y is enumerated into F_{s+1}. This is the only enumeration into F.

Verification.

Lemma 6.6. *All $N_{i,j}$ are met.*

Proof. Assume $\hat{S}_i, y_i$ witness $\tilde{S}_i \sqsubseteq^\diamondsuit I$ and $\Theta(i,j)$ holds. Let $\gamma = \alpha^\frown 0 \subset f$ be such that $|\alpha| = 2\langle i,j\rangle + 1$. We build ideals R and Q together as follows: at a given stage s such that $\gamma \subseteq f_s$ and $\hat{s}$ is the most recent stage $< s$ such that $\gamma \subseteq f_{\hat{s}}$ (or 0 if s is the first such stage), for each $x \in P_{\leq s} - P_{\leq \hat{s}}$ put x into R if $x \in F_s$ and put x into Q if $\langle x\rangle \cap F_s = 0$. It is straightforward to see that R and Q are in fact complements, so in particular R is a $\diamondsuit$-c subideal of F. To show $F \subseteq^\diamondsuit \hat{S}_i \vee A \vee R$, consider any x which is ever a witness for some requirement (that is, such that subideals of $\langle x\rangle$ might be put into F). If x is a released witness, it is in A. If x is a witness for a lower-priority requirement than $N_{i,j}$ it was chosen to be in $\hat{S}_i$. The only remaining possibility is that x is a permanent witness for a higher-priority requirement, and as there are only finitely-many of those the witnesses generate a principal ideal. □

Lemma 6.7. *If $S \sqsubseteq^\diamondsuit I$ is $\diamondsuit$-nc modulo A, then $(\star)$.*

Proof. For this proof we use the equivalent characterization of $\diamondsuit$-complementation that X is $\diamondsuit$-c modulo A if there is $R \subseteq^\diamondsuit X$ such that $X \subseteq^\diamondsuit A \vee R$, where R is $\diamondsuit$-c modulo 0 in the original sense.

Suppose $\langle i,j\rangle$ is the least pair such that $\hat{S}_i, y_i$ witness $S = \tilde{S}_i \sqsubseteq^\diamondsuit I$, and j witnesses the failure of $(\star)$ for $\tilde{S}_i$. By Lemma 6.6, there is some $\diamondsuit$-c $R \subseteq F$ such that $F \subseteq^\diamondsuit \hat{S}_i \vee A \vee R$. Hence $\tilde{S}_i \cap F \subseteq^\diamondsuit \hat{S}_i \vee A \vee R$, and since $\tilde{S}_i \cap \hat{S}_i \subseteq \langle y_i\rangle$, in fact $\tilde{S}_i \cap F \subseteq^\diamondsuit A \vee R$. By $\neg(\star)$, $\tilde{S}_i \subseteq^\diamondsuit A \vee F$; i.e., up to a principal ideal the portion of $\tilde{S}_i$ outside F is contained in A, so combining with the previous containment we get $\tilde{S}_i \subseteq A \vee R$. We show $\tilde{S}_i \cap R$ witnesses $\tilde{S}_i$ is $\diamondsuit$-c modulo A. Clearly $\tilde{S}_i \cap R \subseteq \tilde{S}_i$, and by the work above $\tilde{S}_i \subseteq^\diamondsuit (\tilde{S}_i \cap R) \vee A$. It remains to show $\tilde{S}_i \cap R$ is $\diamondsuit$-c modulo 0. Let Q be as in Lemma 6.6. Then $Q \vee \hat{S}_i$ witnesses that $\tilde{S}_i \cap R$ is $\diamondsuit$-c mod 0 as follows:

$$(Q \vee \hat{S}_i) \cap \tilde{S}_i \cap R = (Q \cap \tilde{S}_i \cap R) \vee (\hat{S}_i \cap \tilde{S}_i \cap R) =^\diamondsuit 0 \vee 0,$$

because Q and R, and $\tilde{S}_i$ and $\hat{S}_i$, are $\diamondsuit$-complements.

$$(Q \vee \hat{S}_i) \vee (\tilde{S}_i \cap R) = (Q \vee \hat{S}_i \vee \tilde{S}_i) \cap (Q \vee \hat{S}_i \vee R) =^\diamondsuit (Q \vee I) \cap M,$$

the second half again because Q and R are complements; $Q \vee I =^\diamondsuit M$ because $R \subseteq F \subseteq I$. □

Lemma 6.8. *If $(\star)$ holds for $\tilde{S}_i \sqsubseteq^\diamondsuit I$, then all $P_{i,j,k}$ are met, and hence $\tilde{S}_i \cap F$ is $\diamondsuit$-nc mod A.*

Proof. Let $\beta = \delta^\frown 0 \subset f$ where $|\delta| = 2\langle i,j,k\rangle$. We first show that if $(\star)$ holds for $\tilde{S}_i$, then after some stage s, β has a permanent witness. By induction, let t be a stage after which β is never again initialized, and such that if $\gamma \subset \beta$ or $\gamma <_L \beta$ and γ ever has a permanent witness, it has one by stage t. Define the set $G = \{\alpha : \alpha^\frown 0 \subseteq \beta \;\&\; |\alpha| = 2\langle i_\alpha, j_\alpha\rangle + 1\}$. G is the set of nodes corresponding to higher-priority requirements for which ($\star$) fails. Since there are only finitely-many such nodes, ($\star$) also fails for $\bigcup\{\tilde{S}_{i_\alpha} : \alpha \in G\}$. Since ($\star$) holds for $\tilde{S}_i$ (in particular, i is not one of the i_α), it must also hold for $\tilde{S}_i \cap \bigcap\{\hat{S}_{i_\alpha} : \alpha \in G\}$. Hence β will eventually acquire a permanent witness.

Given such a witness x, as we see subideals of $\langle x\rangle$ enter $\hat{R}_j$ we feed them into F. Any other requirement with a witness comparable to x will have the same i, j values and hence be enumerating according to the same criterion, so any portion of $\langle x\rangle$ that does not enter $\hat{R}_j$ stays out of F. Since $x \notin \langle z_j, m_k\rangle$, there is some $y \in \langle x\rangle$ such that $\langle y\rangle \cap \langle z_j, m_k\rangle = 0$. If y is in $\tilde{R}_j$, it will show $\tilde{R}_j \not\subseteq (\tilde{S}_i \cap F) \vee \langle m_k\rangle$; if it is in $\hat{R}_j$ it will show $\tilde{S}_i \cap F \not\subseteq \tilde{R}_j \vee A \vee \langle m_k\rangle$. As $\tilde{S}_i \cap F$ has no witnesses to $\diamondsuit$-complementation mod A, it must be noncomplemented. □

This completes the proof of the theorem. □

We use the uniformity of Theorem 6.5 to make the following definition.

Definition 6.9. For an ideal A with a true stage enumeration, let $\hat{e}$ denote the computable function such that for all e, $I_{\hat{e}} = F_e$, where F_e is as defined in Theorem 6.5.

The following two proofs directly translate the proofs of Theorem 8.2 and Corollary 8.3 in [6], where Cholak and Harrington's Theorem 7.3 is our 6.5 and their Theorem 4.1 is our 6.1. They are included for readability.

Theorem 6.10. *Suppose that A has a true stage enumeration. $\mathcal{P} \in \mathcal{J}_{A,\vec{U}}$ if and only if there is an e and a $\diamondsuit$-split S of I_e such that there are a nonprincipal collection of $x = m_t$ and stages $s > t$ where s is a true stage, $x \in S$, $\langle x\rangle \cap I_{\hat{e},s} = 0$, $x \notin A_s$, and $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}_{\hat{e}}, S \cap I_{\hat{e}})$. Hence for any non-high c.e. ideal A, the statement "$\mathcal{P} \in \mathcal{J}_{A,\vec{U}}$" is Σ^A_3.*

Proof. If $\mathcal{P} \in \mathcal{J}_{A,\vec{U}}$, there is some c.e. ideal I_e such that for all $\vec{B}$ there is some $S \sqsubseteq^\diamondsuit I_e$ that is $\diamondsuit$-nc modulo A and such that $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}, S)$. Let $\vec{B}$ be $\vec{B}_{\hat{e}}$ and S the appropriate $\diamondsuit$-nc $\diamondsuit$-split of I_e. Then by Theorem 6.5, there are a nonprincipal collection of $x = m_t$ and stages $s > t$ where s is a true stage, $x \in S$, $\langle x\rangle \cap I_{\hat{e},s} = 0$, and $x \notin A_s$. That $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}_{\hat{e}}, S \cap I_{\hat{e}})$ holds follows by Lemma 5.4 from the fact that $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}_{\hat{e}}, S)$ holds.

Now assume there is a $\diamondsuit$-split S of I_e such that there are a nonprincipal collection of $x = m_t$ and stages $s > t$ where s is a true stage, $x \in S$, $\langle x \rangle \cap I_{\hat{e},s} = 0$, $x \notin A_s$, and $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}_{\hat{e}}, S \cap I_{\hat{e}})$. By Theorem 6.5 $S \cap I_{\hat{e}}$ is $\diamondsuit$-nc mod A; note $S \cap I_{\hat{e}}$ is a $\diamondsuit$-split of $I_{\hat{e}}$, and so by Theorem 6.1, $\vec{U}, A$, and $I_{\hat{e}}$ realize $\mathcal{P}$, so $\mathcal{P} \in \mathcal{J}_{A,\vec{U}}$. □

Corollary 6.11. *For any $A \in G$, the statement "$\mathcal{P} \in \mathcal{J}_{A,\vec{U}}$" is Σ_3^A.*

Proof. By Theorem 6.10 we need only worry about A that are high. By Corollary 6.2, "$\mathcal{P} \in \mathcal{J}_{A,\vec{U}}$" is Σ_4^0 and hence $\Sigma_2^{\emptyset''}$. By assumption on A this is $\Sigma_2^{A'}$, and hence "$\mathcal{P} \in \mathcal{J}_{A,\vec{U}}$" is Σ_3^A. □

7. Realizing patterns selectively

The following theorem, which is the heart of the proof, corresponds to Theorem 6.7 in [6]. We will prove it all at once instead of building up theorems and corollaries as Cholak and Harrington do, to confirm in this different structure that everything works out. We note that this entire construction is uniform; [6] Corollary 6.1 adds nonuniformity via a second strategy, but even there it is not necessary to obtain the generalization (all C instead of just $C \subseteq W$).

Recall Definition 5.3, that $\mathcal{J}_{A,\vec{U}}$ is the set of special $\mathcal{L}$-patterns $\mathcal{P}$ such that $\exists Y(\vec{U}, A, Y$ realize $\mathcal{P})$.

Theorem 7.1. *Given the patterns $\mathcal{P}_i$ defined in §5, a c.e. ideal F, and a Σ_3^F ideal J, there exist a c.e. ideal $A \equiv_T F$ and $\mathcal{L}$-interpretation $\vec{U}$ such that $m_i \in J \iff \mathcal{P}_i \in \mathcal{J}_{A,\vec{U}}$.*

The rest of the section will be a proof of the theorem. Define $\widetilde{W}$ as in Proposition 4.4 with degree $\mathbf{0}$ and $Z = M$; recall $\widetilde{W}$ is then $\diamondsuit$-nc modulo any $I \neq^{\diamondsuit} M$. Designate some principal subideal of $\widetilde{W}$ as W_{-1}. Using the $\diamondsuit$-Friedberg splitting theorem, split $\widetilde{W} \cap \overline{W_{-1}}$ uniformly into $W_{\langle i,n \rangle}$ for $i, n \in \omega$.

Within W_{-1} we code F into A so that $F \leq_T A$. Let $b : M \to W_{-1}$ be a computable isomorphism, and let $b(x) \in A$ if and only if $x \in F$.

The role of the remaining splits of $\widetilde{W}$ is to ensure there is a Y such that $\vec{U}, A$, and Y realize $\mathcal{P}_i$ for $m_i \in J$, and to diagonalize against all such Y for $m_i \notin J$. Permitting will ensure $A \leq_T F$. Since J is Σ_3^F, and "is Ψ_n^F total?" is Π_2^F-complete, we can construct a uniform sequence of functionals $\Psi_{\langle i,n \rangle}^F$ such that $i \in J \iff \exists n(\Psi_{\langle i,n \rangle}^F$ is total). We use the totality of $\Psi_{\langle i,n \rangle}^F$ to determine the behavior of the construction within $W_{\langle i,n \rangle}$. If $\Psi_{\langle i,n \rangle}^F$ is partial, we will make $A \cap W_{\langle i,n \rangle} =^{\diamondsuit} W_{\langle i,n \rangle}$. If $\Psi_{\langle i,n \rangle}^F$

is total, within $W_{\langle i,n\rangle}$ we build $\vec{U}_{\langle i,n\rangle}$ with all its components contained in $W_{\langle i,n\rangle}$ such that $\vec{U}_{\langle i,n\rangle}, A$, and $W_{\langle i,n\rangle}$ realize $\mathcal{P}_i$, and such that for all c.e. ideals C and all $j \neq i$, $\vec{U}_{\langle i,n\rangle}$, A, and C do not realize $\mathcal{P}_j$. We will show this gives the necessary global properties for $\vec{U}$ and A.

7.1. **Building within $\boldsymbol{W_{\langle i,n\rangle}}$.** In this subsection we drop the subscript $\langle i,n\rangle$ from W, $\vec{U}$, and Ψ^F.

Theorem 7.2. *Let $\widetilde{W}$ be a c.e. ideal built as in Proposition 4.4 (in particular, $\widetilde{W}$ is $\diamondsuit$-nc modulo all $I \neq^{\diamondsuit} M$), W a (fixed) $\diamondsuit$-Friedberg split of $\widetilde{W}$, and S any $\diamondsuit$-Friedberg split of W. Let Ψ^F be a functional. Effectively in i, W, and a $\mathcal{B}$-interpretation $\vec{B}_S$, there exist A, $\vec{U}$ and $\vec{D}$ such that*

(i) *if Ψ^F is total, $\vec{D}$ witnesses $\varphi_{\mathcal{P}_i}(A, \vec{U}, \vec{B}_S, S)$, W and S are $\diamondsuit$-nc modulo A, and all components of $\vec{U}$ and $\vec{D}$ are contained in S.*
(ii) *if Ψ^F is partial, $A =^{\diamondsuit} W$.*

Furthermore, for all $j \neq i$ and for all c.e. ideals C, the sets $\vec{U}$, A, and C do not realize $\mathcal{P}_j$ (this is automatic in (ii)*). Moreover, a witness $\vec{B}_{j,C}$ to nonrealization can be found effectively from j and C, with all components contained in C.*

We set the usual convention on the functional $\Psi^F(e)$, that its use $\psi_s(e)$ is nondecreasing in s and e.

Proof. Of course we cannot know at any finite stage whether Ψ^F is total, so we act as though it is. For S and W we must meet the requirements

$$R_{\langle p,q\rangle} : \text{for } X \in \{S, W\}, (I_p \searrow X) \not\subseteq A \vee \langle m_q\rangle.$$

By Lemma 4.8, these requirements ensure W and S are $\diamondsuit$-nc modulo A. We meet $R_{\langle p,q\rangle}$ by waiting for a ball in $(I_p \searrow X) \cap \overline{\langle m_q\rangle}$ and restraining it out of A.

For $\vec{B}_S$ and S we have one global requirement:

$$Q : \text{build } \vec{U} \text{ and } \vec{D} \text{ such that } \vec{D} \text{ witnesses } \varphi_{\mathcal{P}_i}(A, \vec{U}, \vec{B}_S, S).$$

We also have local negative requirements for each $j \neq i$ and c.e. ideal C:

$$N_{j,C} : \text{build a } \mathcal{B}\text{-interpretation } \vec{B}_{j,C} \text{ such that } \forall \tilde{S} \sqsubseteq^{\diamondsuit} C, \forall \vec{X},$$

$$N_{j,C,\tilde{S},\vec{X}} : \begin{array}{l} \vec{X} \text{ does not witness } \varphi_{\mathcal{P}_j}(A, \vec{U}, \vec{B}_{j,C}, \tilde{S}) \\ \text{or } \tilde{S} \text{ is } \diamondsuit\text{-c modulo } A. \end{array}$$

We enumerate only balls from S into the U_i and D_i. To meet Q we simply respond to $\vec{B}_S$ (i.e., to BLUE) to satisfy Definition 5.1 of $\varphi_{\mathcal{P}}$: If x is a ball and $p \in \mathcal{T}$, then if $p \in \mathcal{B}_i$ and $x \in D_{u(p)}$ (where we let $D_{u(b_0)} = S$), then if x enters B_p we put x into D_p, and else not. If $p \in \mathcal{R}_i$ and $x \in D_p$ then x must enter D_q for some $q \in d(p)$, but we get to choose which one. If $p \in \ell(j)$ for some j and $x \in D_p$ we must put x into U_j.

We meet $N_{j,C,\tilde{S},\vec{X}}$ by manipulating the movement of balls on $\mathcal{P}_i$ and $\mathcal{P}_j$ within our constraints. We control the RED moves on $\mathcal{P}_i$ via $\vec{D}$ and the BLUE moves on $\mathcal{P}_j$ via $\vec{B}_{j,C}$ (which forces $\vec{X}$'s hand). RED controls his own moves on $\mathcal{P}_j$ via $\vec{X}$ and BLUE controls his own moves on $\mathcal{P}_i$ via $\vec{B}_S$.

For all balls on all patterns, our default move is down and left, unless required otherwise to meet a negative requirement as described Lemma 7.4 below. Our goal is to get $\vec{X}$ to want some ball x to be in a different U-ideal from the one that Q has put it into. We will show in Lemma 7.3 that we can get a permanent witness x for any $N_{j,C,\tilde{S},\vec{X}}$ such that $\tilde{S}$ is $\diamondsuit$-nc, and in Lemma 7.4 that with such an x we can make $\vec{X}$ fail to witness $\varphi_{\mathcal{P}_j}(A, \vec{U}, \vec{B}_{j,C}, \tilde{S})$.

7.1.1. *Construction nodes and their duties.* Before giving the construction, we define *restraint*: if β sets restraint on x, not only x but all elements of $\langle x\rangle$ and all y such that $x \in \langle y\rangle$ are restrained with priority β from being enumerated into A; note that this is a principal collection. Restraint is only set once; if later another node wishes to use x for a different purpose, the restraint on x is still priority β (this is not needed for the present theorem, but will be used to obtain $A \leq_T F$ for Theorem 7.1).

The organization of the construction is on the tree $2^{<\omega}$. Fix a listing of all tuples $(j, C, \tilde{S}, \hat{S}, z, \vec{X})$ where $j \neq i$, $C, \tilde{S}$, and $\hat{S}$ are c.e. ideals, $z \in M$, and $\vec{X}$ is an interpretation of $\mathcal{P}_j$ over $\vec{U}$. Assign all $\alpha \in 2^{<\omega}$ of length $2e$ to the e^{th} such tuple. It is Π^0_2 to say whether $\hat{S}$, z witness $\tilde{S}$ is a split of C, so we use that as our criterion for putting $\alpha^\frown 0$ on the true path. Define a "length of agreement" to that end:

$$\begin{aligned}\ell(\alpha, s) = \max\{y : (\forall x < y)[m_x \in C_s \rightarrow \\ m_x \in \langle z\rangle \ \vee \ m_x \in (\tilde{S}_s \vee \hat{S}_s) - (\tilde{S}_s \cap \hat{S}_s)]\}.\end{aligned}$$

The first time f_s gets to extend α it goes to $\alpha^\frown 0$, and afterward it only goes to $\alpha^\frown 0$ if s is α-expansionary. Then $\alpha^\frown 0 \subset f$ (for $\alpha = f \restriction (2e)$) $\iff \liminf_s \ell(\alpha, s) = \infty \iff \hat{S}$, z witness $\tilde{S}$ is a $\diamondsuit$-split of C.

Each odd-length node has multiple duties. Suppose β is accessible at stage s, where $|\beta| = 2e+1$. If $\Psi^F(e)[s]\uparrow$, halt the stage. Otherwise, for $e = \langle p, q\rangle$, if a ball x enters $(I_p \searrow W) \cap \overline{\langle m_q\rangle}$ at stage s and β does not already have another such ball restrained, set A-restraint with priority β on the $\lhd$-least such x, if it is not already restrained. Likewise β may set restraint on a ball in $(I_p \searrow S) \cap \overline{\langle m_q\rangle}$.

If the final bit of β is 0 it will furthermore try to get balls for its collection, to use to satisfy $N_\beta := N_{j,C,\tilde{S},\vec{X}}$. Each β will have an "interval" $P_{\leq u_\beta} - P_{\leq d_\beta}$ containing the balls N_β might be allowed to use as witnesses. If this interval is not defined, assign β a large fresh number d_β and an even larger number $u_\beta(s)$ and halt the stage. If the interval is defined, we have two cases. If β has no ball in its collection, attempt to get a ball as described below in §7.1.3. If that attempt is successful, or β already had a ball, let $u_\beta(s) = u_\beta(s-1)$ and let $\beta^\frown 1 \subseteq f_s$; set restraint, if not already set. If the attempt is not successful, increase u_β and let $\beta^\frown 0 \subseteq f_s$.

7.1.2. *Override of restraint.* At the end of the stage, any unrestrained elements of W are enumerated into A. As usual, nodes are initialized whenever the true path approximation passes to their left or terminates above them. To begin the next stage, for each e and $|\beta| = 2e+1$, if $F_{s+1} \restriction \psi_s(e) \neq F_s \restriction \psi_s(e)$, enumerate all balls restrained with priority β into A.

It is clear from the construction and the convention on use that if $\Psi^F(e)\uparrow$ infinitely-many times, we have $A =^\diamond W$.

7.1.3. *Obtaining a ball for β's collection.* Note that since $u_\beta(s)$ is nondecreasing and $\beta^\frown 0 \subseteq f_s$ whenever u_β increases at s, if $\beta \subset f$ then $\beta^\frown 0 \subset f$ if and only if $\lim_s u_\beta(s) = \infty$, which is if and only if there is no permanent ball in β's collection.

Define the set

$$G_\beta = \{\gamma : |\gamma| \text{ odd } \& \ (\gamma^-)^\frown 0 = \gamma \subset \beta \ \& \ C_\gamma = C_\beta \ \& \ j_\gamma = j_\beta\},$$

predecessors of β that play analogous roles for other $N_{j,C}$ requirements. For $i = 0, 1$, let $G^i_\beta = \{\gamma : \gamma \in G_\beta \ \& \ \gamma^\frown i \subseteq \beta\}$. For $\gamma \in G^0_\beta$, let $L_\gamma = \hat{S}_\gamma \cap \overline{\langle z_\gamma\rangle}$, and for $\gamma \in G^1_\beta$ let $L_\gamma = \overline{P_{\leq u_\gamma}}$; L_γ contains the elements γ will never use as witnesses ("ever" is from β's perspective; if u_β changes for some γ in G^1_β, β will be initialized). If $G_\beta \neq \emptyset$ let $W_\beta = C_\beta \cap \bigcap\{L_\gamma : \gamma \in G_\beta\}$, and otherwise let $W_\beta = C_\beta$.

We take the ball x for β's collection under the following conditions:

- x enters $P_{\leq u_\beta} - P_{\leq d_\beta}$ at stage s;
- x enters W_β at stage $t \geq s$;

- x is in $\tilde{S}_\beta$, where here stage of entry is unimportant;
- x is not in A (yet).

We may take any such ball, which may result in theft from a lower-priority requirement (but only one with a different j or C).

7.1.4. *Verification.* Assume for Lemmas 7.3 and 7.4 that Ψ^F is total. In that case, for any e there is a stage after which $\Psi^F(e)$ never diverges and hence does not cause initialization.

Lemma 7.3. *If $\beta = \alpha^\frown 0 \subset f$ never gets a permanent witness, $\tilde{S}_\beta$ is $\diamondsuit$-c mod A.*

Proof. Assume β never gets a permanent witness, meaning $\beta^\frown 0 \subset f$ (i.e., $u_\beta \to \infty$). By induction, assume $\tilde{S}_\gamma$ is $\diamondsuit$-c mod A for all $\gamma \in G^0_\beta$, so $\bigcup\{\tilde{S}_\gamma : \gamma \in G^0_\beta\}$ is a $\diamondsuit$-split of C_β that is $\diamondsuit$-c mod A: its $\diamondsuit$-splitting partner is W_β, witnessed by $\bigvee\{\langle z_\gamma\rangle : \gamma \in G^0_\beta\} \vee \bigvee\{P_{\leq u_\gamma} : \gamma \in G^1_\beta\}$, and its $\diamondsuit$-complement is the intersection $\bigcap\{R_\gamma : \gamma \in G^0_\beta\}$ of the $\diamondsuit$-complements of the $\tilde{S}_\gamma$, witnessed by the join of their witnesses. It is enough to show $W_\beta \cap \tilde{S}_\beta$ is $\diamondsuit$-c mod A, because $\tilde{S}_\beta \cap \bigvee\{\tilde{S}_\gamma : \gamma \in G^0_\beta\}$ is $\diamondsuit$-c mod A by $\bigcap\{R_\gamma : \gamma \in G^0_\beta\} \vee \hat{S}_\beta$.

Assume d_β is never reset after stage t, and for all δ, if $\delta <_L \beta$ or $\delta \subset \beta$ and δ ever has a permanent member, it has one by stage t (so no one steals from β at or after stage t). Let $x \notin P_{\leq d_\beta}$ and wait for a least stage $s \geq t$ such that $x \in P_{\leq u_\beta(s)}$ (by assumption, $u_\beta \to \infty$). If x enters W_β it must later enter $\hat{S}_\beta$ or A; otherwise we would take it for β's collection and have a permanent member. Hence $x \in (W_\beta \cap \tilde{S}_\beta) - A$ if and only if $x \in ((W_\beta \cap \tilde{S}_\beta) - A)[s]$. To demonstrate $W_\beta \cap \tilde{S}_\beta$ is $\diamondsuit$-c mod A, at the same stage s if $x \notin ((W_\beta \cap \tilde{S}_\beta) - A)[s]$ take all $y \in \langle x\rangle$ such that $\langle y\rangle \cap W_\beta \cap \tilde{S}_\beta[s] = 0$ and put y into an ideal Z. Z is a $\diamondsuit$-complement of $W_\beta \cap \tilde{S}_\beta$ mod A, witnessed by $P_{\leq d_\beta} \vee \bigvee\{\langle z_\gamma\rangle : \gamma \in G^0_\beta\} \vee \bigvee\{P_{\leq u_\gamma} : \gamma \in G^1_\beta\}$. □

Lemma 7.4. *If β has a permanent witness x, $N_\beta = N_{j,C,\tilde{S},\vec{X}}$ is satisfied via $\vec{X}$ failing to witness $\varphi_{\mathcal{P}_j}(A, \vec{U}, \vec{B}_{j,C}, \tilde{S})$.*

Proof. This proof is identical to that in [6] §5.1.6–10, presented in a condensed form below. If x is in C but not S, β will win with the default "down and left" strategy, because Q will not put x in any U-set, but $\vec{X}$ will want it in U_3. However, we cannot know that a ball is not in S, so we always move x according the four cases below. Multiple iterations may be required; e.g., when x is in r^ℓ_1 on $\mathcal{P}_j$ we may have to move it left from multiple b_2-nodes in $\mathcal{P}_i$ to ensure $\vec{X}$ and Q want x in different U-sets. To say a ball is "in" a node p on $\mathcal{P}_i$ or $\mathcal{P}_j$ means it is in D_p or X_p, respectively. To say a ball is "targeted for" a node p on $\mathcal{P}_j$ means it is in $\mathcal{B}_{j,C,p}$, hereafter referred to as $\mathcal{B}_p$: all B ideals below are elements of $\vec{B}_{j,C}$, not $\vec{B}_S$. As a reminder, the Q strategy obeys $\vec{B}_S$ in building $\vec{D}$ and $\vec{U}$; on $\mathcal{P}_i$ our only control is the direction of descent from b_2^k nodes. On $\mathcal{P}_j$ we build the BLUE strategy $\vec{B}_{j,C}$, which $\vec{X}$ must respect, and hence have control over essentially everything *except* the direction of descent from b_2^k nodes. We do not remark on winning via $\vec{X}$ failing to respect $\vec{B}_{j,C}$.

(1) x is below b_0 on $\mathcal{P}_i$ while it is still in b_0 on $\mathcal{P}_j$.

Strategy: Move x in $\mathcal{P}_j$ opposite to how it moved in $\mathcal{P}_i$. In this case, on $\mathcal{P}_i$ x will be in either b_1 or in some node b_2^0 or below. If it is in b_1 Q puts it in U_0, so enumerate x into $B_{b_2^1}$ to force $\vec{X}$ to enumerate x into $X_{b_2^1}$; $\vec{X}$ will then want x either outside all U sets or in one different from U_0. If x is in b_2^0 or below, Q will not enumerate it into U_0, so enumerate it into B_{b_1} to force $\vec{X}$ to enumerate it into X_{b_1} and want it in U_0.

(2) x is in b_2^ℓ on $\mathcal{P}_i$ and not also in or targeted for $b_2^{\ell'}$ in $\mathcal{P}_j$ for some $\ell' \leq j$.

Strategy: If x is in b_0 on $\mathcal{P}_j$, we are in case (1). If x is in b_1, b_5, b_4, or $b_3^{\ell'}$ on $\mathcal{P}_j$, $\vec{X}$ will never want it in U_1, so we enumerate x into $D_{r_1^\ell}$ so Q will put it into U_1. If x is in $X_{r_1^{\ell'}}$, so $\vec{X}$ wants it in U_1, enumerate x into $D_{r_2^\ell}$ to keep it out of U_1. If x is in $X_{r_2^{\ell'}}$, enumerate it into $B_{b_3^{\ell'}}$ so $\vec{X}$ will be forced to add it to $X_{b_3^{\ell'}}$ and hence want it to be in U_2. Enumerate x into $D_{r_1^\ell}$ so Q will enumerate it into U_1.

(3) x is in r_2^ℓ in $\mathcal{P}_j$ (the remaining "internal" node type, which we control descent from) and not also in $r_2^{\ell'}$ in $\mathcal{P}_i$ (which we do not control descent from).

Strategy: If x is in b_1, $r_1^{\ell'}$, b_4, or b_5 on $\mathcal{P}_i$, and hence Q wants it out of U_2, enumerate x into $B_{b_3^\ell}$ so $\vec{X}$ will want it in U_2. If $x \in D_{b_2^{\ell'}}$, enumerate it into both $B_{b_3^\ell}$ and $D_{r_1^{\ell'}}$ so $\vec{X}$ wants it in U_2 and Q wants it in U_1. If $x \in D_{b_3^{\ell'}}$, enumerate x into $B_{b_2^{\ell+1}}$ (which is b_4 if $\ell = j$); if this is b_4 we are done because $\vec{X}$ wants x in U_3, if anywhere, and Q wants x in U_2.

If x is in D_{b_0}, enumerate it into $B_{b_3^\ell}$ and wait; if x later enters $D_{b_2^0}$ (by the action of BLUE) enumerate it into $D_{r_1^0}$ so $\vec{X}$ and Q will want it in U_2 and U_1, respectively.

(4) Not case (1)–(3).

Strategy: In this case we know that x starts on $\mathcal{P}_j$ before $\mathcal{P}_i$, whenever $x \in X_{r_2^\ell}$ x is also in $D_{r_2^{\ell'}}$ for some $\ell' \leq i$, and whenever $x \in D_{b_2^\ell}$ either $x \in X_{b_2^{\ell'}}$ or $x \in B_{b_2^{\ell'}}$ (and hence is targeted for $b_2^{\ell'}$) for some $\ell' \leq j$. We assume all along that we do not win in an easy way by RED and BLUE failing to coordinate their attacks against us.

That x starts on $\mathcal{P}_j$ first means x enters $B_{b_2^0}$ before x enters $D_{b_2^0}$. Then x enters $D_{r_2^0}$ before x enters $X_{r_2^0}$, and x enters $B_{b_2^1}$ before x enters $D_{b_2^1}$. By induction this continues until we hit the end of one of the patterns; i.e., until x enters $D_{b_2^{k+1}}$, $k = \min\{i, j\}$.

If $k = j$, $B_{b_2^{k+1}} = B_{b_4}$, the second-to-last node in $\mathcal{P}_j$, but $D_{b_2^{k+1}} \neq D_{b_4}$. We are in case (2) above, after all.

If $k = i$, $D_{b_2^{k+1}} = D_{b_4}$, the second-to-last node in $\mathcal{P}_i$, and $B_{b_2^{k+1}} \neq B_{b_4}$. Q can only add x to U_3, so we make sure $\vec{X}$ wants x in U_1 or U_2 instead: we know RED must move x downward in $\mathcal{P}_j$. If RED moves x right into $X_{r_1^{k+1}}$ $\vec{X}$ wants x in U_1. If RED moves x left into $X_{r_2^{k+1}}$ we take over and enumerate x into $B_{b_3^{k+1}}$ to force it into $X_{b_3^{k+1}}$ and hence make $\vec{X}$ want x in U_2. □

Lemma 7.5. *All $R_{\langle p,q\rangle}$ are met.*

Proof. As in the original Friedberg Splitting Theorem, if X is a $\diamondsuit$-Friedberg split of Y and $I_e \searrow Y \neq^\diamondsuit 0$, then $I_e \searrow X \neq^\diamondsuit 0$ also. Since $\widetilde{W}$ is $\diamondsuit$-nc mod A, we must have $I_e \searrow \widetilde{W} \neq^\diamondsuit 0$ for all $I_e \neq^\diamondsuit 0$ (see Lemma 4.6) and hence the same for S and W (the removal of the principal ideal W_{-1} does not impact this). Since no balls in $I_p \searrow X$ are off-limits to $R_{\langle p,q\rangle}$ except those in $\langle m_q\rangle$ and those enumerated into A while $\Psi^F \restriction (\langle p,q\rangle + 1)$ settles – a principal collection – $R_{\langle p,q\rangle}$ will eventually find the permanent witnesses it needs. □

Since Lemma 7.3 shows that $N_{j,C,\tilde{S},\vec{X}}$ gets a permanent witness x unless it is satisfied by $\tilde{S}$ being $\diamondsuit$-c mod A, Lemma 7.4 shows we can win $N_{j,C,\tilde{S},\vec{X}}$ with x while meeting the global requirement Q, and Lemma 7.5 shows all $R_{\langle p,q\rangle}$ are met, this completes the proof of the theorem. □

Recall that Theorem 6.1 constructs a BLUE strategy $\vec{B}_e$, for e indexing $\vec{U}$, $\mathcal{P}$, A, and W, such that the existence of a $\diamondsuit$-split S of W such that S is $\diamondsuit$-nc modulo A and $\varphi_{\mathcal{P}}(A, \vec{U}, \vec{B}_e, S)$ is equivalent to the realization of $\mathcal{P}$ by $\vec{U}, A$, and W. All theorems at hand are uniform, and hence we may apply Theorem 6.1 in each split $W_{\langle i,n\rangle}$, using $\vec{B}_e$

for e indexing $\vec{U}_{\langle i,n\rangle}$, $\mathcal{P}_i$, $A_{\langle i,n\rangle}$, and $W_{\langle i,n\rangle}$, where e is obtained by the Recursion Theorem. We obtain the following result.

Corollary 7.6. *Let $\widetilde{W}$ be a c.e. ideal built as in Proposition 4.4, and W a $\diamondsuit$-Friedberg split of $\widetilde{W}$. Let Ψ^F be a functional. Effectively in i and W, there exist A and $\vec{U}$ such that*

(i) *if Ψ^F is total, $\vec{U}$, A, and W realize $\mathcal{P}_i$ (and all components of $\vec{U}$ are contained in W).*
(ii) *if Ψ^F is partial, $A =^{\diamondsuit} W$.*

Furthermore, for all $j \neq i$ and for all c.e. ideals C, the sets $\vec{U}$, A, and C do not realize $\mathcal{P}_j$. Moreover, a witness $\vec{B}_{j,C}$ to nonrealization can be found effectively from j and C, with all components contained in C.

7.2. **Finishing up.** A and $\vec{U}$ are the join of their component pieces constructed within each $W_{\langle i,n\rangle}$, and within W_{-1} in the case of A.

Lemma 7.7. *$A \leq_T F$.*

Proof. $\widetilde{W}$ is computable and contains A. If $x \in \widetilde{W}$, x will eventually enter some $W_{\langle i,n\rangle}$ or W_{-1}. Within W_{-1}, A is Turing-equivalent to F. Within $W_{\langle i,n\rangle}$, by §7.1.2, if x enters at stage s either it is put into A immediately, or it is restrained out of A with priority β, where $|\beta| = 2e+1$. In that case, x enters A if and only if $F \upharpoonright \psi_s(e)$ changes after stage s. □

Lemma 7.8. *For all i, $m_i \in J \iff \exists Y(\vec{U}, A, Y$ realize $\mathcal{P}_i)$.*

Proof. As all components of $\vec{U}$ are empty on W_{-1}, realization is not affected by how we construct A within W_{-1}.

Note that when working within a single $W_{\langle j,n\rangle}$, A and $A_{\langle j,n\rangle}$ may be exchanged without affecting either $\diamondsuit$-(non)complementation (by $W_{\langle j,n\rangle}$ being a split) or realization.

If $m_i \in J$, then for some n, $\Psi^F_{\langle i,n\rangle}$ is total, and by Corollary 7.6, $\vec{U}_{\langle i,n\rangle}, A_{\langle i,n\rangle}$, and $W_{\langle i,n\rangle}$ realize $\mathcal{P}_i$. Hence, by Lemma 5.4, $\vec{U}, A$, and $W_{\langle i,n\rangle}$ realize $\mathcal{P}_i$ and $W_{\langle i,n\rangle}$ is the required Y.

Now suppose $m_i \notin J$, and suppose for some Y, $S \sqsubseteq^{\diamondsuit} Y$ is $\diamondsuit$-nc mod A and witnesses that $\vec{U}, A$, and Y realize $\mathcal{P}_i$. By Lemma 4.5, $S \cap \widetilde{W}$ is also $\diamondsuit$-nc mod A, and since W_{-1} is principal there must be at least one $\langle j,n\rangle$ such that $S \cap W_{\langle j,n\rangle}$ is also $\diamondsuit$-nc mod A. Note $S \cap W_{\langle j,n\rangle} \sqsubseteq^{\diamondsuit} Y \cap W_{\langle j,n\rangle}$. Again by Lemma 5.4, $\vec{U}, A, Y \cap W_{\langle j,n\rangle}$ realize $\mathcal{P}_i$ with this $S \cap W_{\langle j,n\rangle}$. However, if $\Psi^F_{\langle j,n\rangle}$ is partial, this contradicts the fact that $A \cap W_{\langle j,n\rangle} =^{\diamondsuit} W_{\langle j,n\rangle}$ and hence $S \cap W_{\langle j,n\rangle}$ must be $\diamondsuit$-c mod A. If $\Psi^F_{\langle j,n\rangle}$ is total (necessitating $j \neq i$), since $U_\ell \cap W_{\langle j,n\rangle} = \vec{U}_{\langle j,n\rangle,\ell} \cap W_{\langle j,n\rangle}$, this contradicts Corollary 7.6 again, which says $\vec{U}_{\langle j,n\rangle}$, A, and $Y \cap W_{\langle j,n\rangle}$ do *not* realize $\mathcal{P}_i$. □

8. Definability and the main result

Finally we put together the defining formula.

Definition 8.1. Let $\varphi_J(A)$ be the $\mathcal{L}_{\omega_1,\omega}$ sentence "there exists an $\mathcal{L}$-interpretation $\vec{U}$ such that $(\forall i)[m_i \in J \iff \mathcal{P}_i \in \mathcal{J}_{A,\vec{U}}]$."

By Theorem 7.1, for all c.e. ideals F and all ideals $J \in \Sigma^F_3$ there is a c.e. ideal $A \equiv_T F$ such that $\varphi_J(A)$. Conversely, by Corollary 6.11, if $\varphi_J(A)$ holds J is Σ^A_3. As in [6], the coding is closed upward in the sense that if $B \geq_T A$ and $\varphi_J(A)$ there is $C \equiv_T B$ such that $\varphi_J(C)$.

Since for each fixed $\mathcal{P}$, the statement $\mathcal{P} \in \mathcal{J}_{A,\vec{U}}$ is $\{\subseteq\}$-definable, $\varphi_J(A)$ is invariant under automorphisms of G. Once again, $\varphi_J(A)$ is an $\mathcal{L}(A)$ property.

Theorem 8.2. *Let $\mathcal{C} = \{\deg(J) : J$ is a Σ^0_3 ideal such that $J \geq_T \mathbf{0''}\}$. Let $\mathcal{D}$ be a subset of $\mathcal{C}$ that is upward closed in $\mathcal{C}$. Then there is $\varphi_{\mathcal{D}}(A)$, invariant under automorphisms of G, such that*

$$(\forall F)[F'' \in \mathcal{D} \iff (\exists A)[\varphi_{\mathcal{D}}(A) \,\&\, A \equiv_T F]].$$

The proof is again a direct translation of the proof of Theorem 8.5 in [6], included for readability of the present paper.

Proof. Let $\varphi_{\mathcal{D}}(A)$ be the infinite disjunction of all $\varphi_J(A)$ where J is a Π^0_3 ideal with degree in $\mathcal{D}$.

If $F'' \in \mathcal{D}$ there is a Π^0_3 set J in the above disjunction such that J is Σ^F_3, and hence, by Theorem 7.1, there is an A such that $A \equiv_T F$ and $\varphi_J(A)$.

Now suppose for some c.e. ideal F there is an $A \equiv_T F$ such that $\varphi_J(A)$ is in the disjunction $\varphi_{\mathcal{D}}(A)$. Then, by Corollary 6.11, J is Σ^A_3, and by definition of $\varphi_{\mathcal{D}}(A)$ it is also Π^0_3. Hence J is $\Delta^A_3 = \Delta^F_3$, so $J \leq_T F''$ and $F'' \in \mathcal{D}$. □

We have the same corollaries, as well, using Corollary 3.6 to move from G to $\mathcal{E}_\Pi$. The first is really the main result of this paper, and so we will call it a theorem.

Theorem 8.3. *For all $n \geq 2$, the* high$_n$ *and* non-low$_n$ *c.e. degrees are invariant over $\mathcal{E}_\Pi$.*

Proof. Let $\mathcal{D} = \{\boldsymbol{a}'' : \boldsymbol{a}$ is a high$_n$ (non-low$_n$) c.e. degree$\}$. □

Corollary 8.4. *If $\mathcal{F}$ is a class of c.e. degrees such that if $\boldsymbol{a} \in \mathcal{F}$ and $\boldsymbol{a''} \leq \boldsymbol{b''}$ then $\boldsymbol{b} \in \mathcal{F}$, then $\mathcal{F}$ is invariant over $\mathcal{E}_\Pi$.*

Proof. Let $\mathcal{D} = \{\boldsymbol{a}'' : \boldsymbol{a} \in \mathcal{F}\}$. □

Corollary 8.5. *If $\boldsymbol{a''} > \boldsymbol{b''}$ then there is some $A \in \boldsymbol{a}$ that is not automorphic to any $B \in \boldsymbol{b}$.*

Proof. Let $\mathcal{D} = \{\boldsymbol{d}'' : \boldsymbol{d} \geq \boldsymbol{a}\}$. There is some $A \in \boldsymbol{a}$ such that $\varphi_{\mathcal{D}}(A)$, but if $B \in \boldsymbol{b}$ then $\neg\varphi_{\mathcal{D}}(B)$, so A is not automorphic to B. □

9. Final notes

9.1. **General notes on translation.** The present paper is the second to use methods from the c.e. sets within G or $G^\diamondsuit$. It seems likely more of the wealth of results for the c.e. sets may be transferred to the Π^0_1 classes in this way, so we set down some rules of thumb.

Most translation is direct, with sets becoming ideals and union becoming join. The proofs in this paper go through nearly identically, modulo the extra parameter in the requirements used to make working mod principal explicit. Significant or nonuniform changes stem from the dependence between elements of ideals and the fact that an ideal need not $\diamondsuit$-split M even if its splitting partner were allowed to be of arbitrary complexity. The dependence of elements means $x \in W$ in $\mathcal{E}$ may stay $x \in W$ or become $\langle x \rangle \cap W \neq 0$ in G, and $x > k$ may become $x \rhd m_k$ or $x \notin P_{\leq k}$. For example, condition (ii) for enumeration into S_β in Theorem 6.1 remains $x \in \tilde{S}_{i,s} \cap W_s$ as it was in Cholak and Harrington [6] Theorem 4.2, because we only care that S_β is a subideal of $\tilde{S}_{i,s} \cap W_s$. However, in the splitting theorems (4.7 and 4.9), to choose where to enumerate the next generating element b of B, we require only $\langle b \rangle \cap I_e \neq 0$; to ensure the splits are disjoint we enumerate only a disjoint generating set for B and therefore cannot guarantee any b is a member of I_e, even for $I_e \neq^\diamondsuit 0$. The fact that ideals need not be $\diamondsuit$-complemented even in the lattice of *all* subideals of M means in Theorem 6.1 and Lemma 6.6 we build two ideals simultaneously to ensure we have a complementary pair, whereas in the corresponding sections of [6] (§4.1.3 and §7.1.5) they simply build one set, computably. It also requires in Theorem 7.1 that W_{-1} be a principal subideal of $\widetilde{W}$ rather than a Friedberg split as it is in [6] §6.5.

9.2. **Open questions.** There are few open questions about degree invariance remaining. Known invariant classes in $\mathcal{E}$ which are not covered by this work are the high degrees, $\{\boldsymbol{0'}\}$, and the d-simple degrees, a definable class that splits the low degrees (Lerman and Soare, [9]). In the reverse direction, the array noncomputable (anc) degrees are known to be invariant in $\mathcal{E}_\Pi$ (Cholak, Coles, Downey and Herrmann [5]) but not in $\mathcal{E}$. As discussed in [13], however, the $G^\diamondsuit$ viewpoint appears inapplicable there: the Π^0_1 classes that give the degree invariance are the perfect thin classes, all of which are $\diamondsuit$-trivial in G.

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