Π^0_1 Class Tutorial Notes

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12. Scott proved any completion of PA F is such that the degrees recursive in F form a basis for Π^0_1 classes. Degrees of completions $\subseteq$ degrees of consistent extensions is clear. Solovay did the rest. See Odifreddi, *Classical Recursion Theory*, p. 510–515.
16. $\boldsymbol{a}$ is hyperimmune-free if each function $f \leq_T \boldsymbol{a}$ is majorized by some computable function g_f. Majorized: $\forall n, f(n) < g(n)$ (as opposed to dominated, which is a.e. n). Nonzero c.e. degrees are all hyperimmune.
19. $\boldsymbol{b} << \boldsymbol{a}$ means every infinite tree $T \subseteq 2^\omega$ computable in $\boldsymbol{b}$ has a path computable in $\boldsymbol{a}$. So degree-related basis theorems are theorems asserting $\mathbf{0} << \boldsymbol{a}$ for some $\boldsymbol{a}$.

    Arslanov: if $\mathbf{0} << \boldsymbol{a}$ and $\boldsymbol{a}$ is a c.e. degree, then $\boldsymbol{a} = \mathbf{0'}$ (so since PA contains only paths $>> \mathbf{0}$, its low and c.e. paths are not the same).

    Kučera: If $\mathbf{0} << \boldsymbol{a} \leq \mathbf{0'}$, then there is a promptly simple degree $\boldsymbol{b} \leq \boldsymbol{a}$ (and hence as p.s. degrees are noncappable (by other c.e. degrees), $\boldsymbol{a}$ must be noncappable).
24. Since the class of randoms has measure 1, all positive-measure Π^0_1 classes must contain randoms. Converse is Kurtz's.

    Schoenfield jump inversion: For any Σ^0_2 set $S \geq_T 0'$ there is a Δ^0_2 set A such that $A' \equiv_T S$. So not only is the jump of a Δ^0_2 set Σ^0_2, *every* properly Σ^0_2 set is the jump of some Δ^0_2 set. Extend by requiring A live in a particular given Π^0_1 class; get as consequence every Σ^0_2 is the jump of some *random* Δ^0_2.
26. Verifications: boil down to leaving dead ends on the tree longer.

    (a) $\Rightarrow$ (b): $X \notin [S] \Rightarrow (\exists n, m)\neg R(m, X \upharpoonright n)$, so for any $k \geq m, n$, $X \upharpoonright k \notin T$, hence $X \notin [T]$.

    (b) $\Rightarrow$ (c): $X \notin [T] \Rightarrow (\exists n, m)\varphi_m(X \upharpoonright n) = 0$, so for any $k \geq m, n$, get $X \upharpoonright k \notin S$, hence $X \notin [S]$.
28. 1-reducible: $B \leq_1 A$ means there is a computable 1-1 function f such that $f(B) \subseteq A$ and $f(\overline{B}) \subseteq \overline{A}$. (i.e., $x \in B \Leftrightarrow f(x) \in A$)
33. Note that the index set for computably continuous functions is itself Π^0_2-complete.
40. The lemma follows from the fact that as we enumerate our ideal we could take each interval as a whole or as the union of several disjoint intervals given by appropriately long successors of the string.
41. Small subsets for the c.e. sets ($\mathcal{E}$): $A \subset_s B$ if $A \subset_\infty B$ and $\forall X, Y \in \mathcal{E}$, $X \cap (B - A) \subseteq Y \Rightarrow Y \cup (X - B) \in \mathcal{E}$.
42. Generally omit finite thin classes from the definition, and then get 'thin or finite' in the part about complementation.
43. Considered as trees, perfect classes are stretched-out copies of $2^{<\omega}$.

    Array noncomputable, original definition: $(\forall e)(\exists n)(W_e \cap F_n = A \cap F_n)$ for some very strong array $\mathcal{F}$. Very strong array: $\mathcal{F} = \{F_n\}_{n \in \omega}$ sequence of nonempty finite sets that partition ω and strictly increase in size, such that there is a computable f with $F_n = D_{f(n)}$ for D the canonical enumeration of finite sets.

    Equivalent, easier definition: c.e. A is anc if for all $g \leq_{wtt} \emptyset'$ there is a function $f \leq_T A$ which is not dominated by g (i.e., infinitely often $f > g$).
45. Theory of true arithmetic is $\mathrm{Th}(\mathbb{N}, +, \times)$. Many-one reducible is 1-reducible as above (28) but f allowed to be non-1-1. An *interpretation* of theory T_1 in theory T_2 is simply a many-one reduction of T_1 to T_2, defined in some natural way on T_1.