Surface Area and Surface Integrals

We've done line integrals, now it's time to generalize a bit more and do surface integrals. Surface area, surface integrals, masses and moments, and the surface integral for flux share the same overall form:

$$\int\int_R (\text{function}) \frac{|\nabla f|}{|\nabla f \cdot p|} dA$$

Let's talk about the parts of the integral common to all the formulas first, and then we'll talk about the different functions that get filled in.

The function f which we are taking the gradient of here is the equation for the surface we are working over. It may be given as something like $z = x^2 + y^2$ or $z = 4 - y^2$, but we must put it into the form $f(x, y, z) = c$ for some constant c. In the examples, we might have $f = z - x^2 - y^2 = 0$ and $f = z + y^2 = 4$, with gradients $-2x\vec{i} - 2y\vec{j} + \vec{k}$ and $2y\vec{j} + \vec{k}$.

The letters p, R, and A all refer to the shadow region of the area we are integrating on. In general, the shadow will be as though a flashlight were shining parallel to one of the coordinate axes. Also, we usually want the shadow to be cast by only one layer of our surface. That may not be totally clear in the abstract, so let's talk about an example. If your surface is a bowl-shaped portion of the paraboloid $z = x^2 + y^2$, then you will want the shadow to be cast on the xy-plane, as though a flashlight were shining down parallel to the z-axis. You do not want the shadow to be cast somewhere parallel to the xz-plane, where two layers are being compressed into one, because then you'd have to deal with the two layers one at a time, instead of doing the whole surface in one fell swoop. In homework problems, if the shadow region is ambiguous it will often be given to you.

The region R is simply the shadow region itself. If you leave figuring out bounds for R until the end, you can often simplify your life through the magic of polar coordinates. The vector p is a unit vector normal to the shadow region – in the example above, we can use k. And finally dA expands out into the two variables defining the plane the shadow region is in or parallel to (you need not have a shadow region always be in one of the axis-defined planes). So in our example, dA is $dxdy$.

The steps for finding that last chunk of the integral, then, are:

(1) Find f by manipulating the given surface equation.
(2) Figure out where the shadow of the surface lies; if it is parallel the v_1v_2-plane then dA is dv_1dv_2.
(3) Find a unit vector p normal to the shadow (usually will be one of the basic vectors $\vec{i}$, $\vec{j}$, $\vec{k}$).
(4) Take f's gradient.
(5) Take the norm of ∇f.
(6) Dot ∇f with p and take absolute value.
(7) Divide (5) by (6).
(8) And finally, once the integral has been totally put together, find bounds for R, the shadow region itself.

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Now we'll talk about the individual versions of this integral. The first, **surface area**, is the simplest – the function in the integral is simply 1. So the steps outlined above are all you need to worry about.

Next we have **surface integrals**. In them, you are integrating some function $g(x, y, z)$ over your surface f. The function there is just g.

For **masses and moments**, the function is the density δ or the density times some appropriate function of one or two variables.

Finally, we have the most complicated one, the **surface integral for flux**. It actually requires some work to set up.

I'm assuming everything has been set up at above, except that bounds for R may not have yet been established. The integral here is

$$\int\int_R (F \cdot n)\frac{|\nabla f|}{|\nabla f \cdot p|}dA$$

where F is the vector field we are finding the flux of, and n is a unit vector giving the direction in which we want to find the flux. F is given, so that's not a problem, but n sometimes seems to come from nowhere. It will always be true that

$$n = \pm\frac{\nabla f}{|\nabla f|}$$

which is the unit vector in the direction of the surface's gradient, or the unit vector opposite to the direction of the surface's gradient. So, then, how do you pick plus or minus? That is an explanation better left to within the examples below. The short version is, "you read the problem and translate the direction it gives into symbols." That means little, though, until you've seen an example or two.

Examples

I'll stick to flux examples, since the others are set up in the same way, barring the choice of n and dotting it with F.

Example 1: Find the flux of the field $F(x, y, z) = 4x\vec{i} + 4y\vec{j} + 2\vec{k}$ outward (away from the z-axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$.

Let's follow the steps from before:

(1) Find f. Well, the surface is $z = x^2 + y^2$, and we need it to be something equal to a constant. To save on negatives, let $f = x^2 + y^2 - z = 0$.
(2) Figure out the shadow. Since we have a bowl sitting on the xy-plane, we'll have the shadow be the circle directly below the bowl, living in the xy-plane. Thus dA will be $dxdy$. Also note that since we cut off at $z = 1$, we have $x^2 + y^2 = 1$ at the top of the bowl, and our shadow is a disk of radius 1, centered at the same point as the bowl: the origin.
(3) Find p normal to the shadow. We need p to be a unit vector normal to the xy-plane, so we let $p = \vec{k}$.
(4) Take the gradient of the surface equation. $\nabla f = 2x\vec{i} + 2y\vec{j} - \vec{k}$.
(5) Take the norm of ∇f. $|\nabla f| = \sqrt{4x^2 + 4y^2 + 1}$.
(6) Dot ∇f with p and take absolute value. $|\nabla f \cdot p| = 1$.
(7) Place the above in a fraction. Since (6) is 1, we get back $\sqrt{4x^2 + 4y^2 + 1}$.
(8) Find bounds for R. We'll leave this till later.

Now we need n. We know n can be

$$n = \pm\frac{\nabla f}{|\nabla f|} = \pm\frac{2x\vec{i} + 2y\vec{j} - \vec{k}}{\sqrt{4x^2 + 4y^2 + 1}}$$

but how do we choose which one? We are told that we want the flux in an outward direction, away from the z-axis. Thinking about what vectors outward from the origin look like in the xy-plane and ignoring the z component for now, we note that if a vector starting at (x, y) points away from the origin, if x is positive, we want the vector's $\vec{i}$ component to be positive, and if negative, we want negative in the vector. The same goes for y - we want our vector to have the same signs for $\vec{i}$ and $\vec{j}$ as are found on x and y in the point we start at. In our equation above, since $2x$ and $2y$ are the coefficients for $\vec{i}$ and $\vec{j}$, we already have what we want without changing the sign,so we choose the positive version and have

$$n = \frac{2x\vec{i} + 2y\vec{j} - \vec{k}}{\sqrt{4x^2 + 4y^2 + 1}}.$$

Finally we dot n with F, giving us

$$F \cdot n = \frac{8x^2 + 8y^2 - 2}{\sqrt{4x^2 + 4y^2 + 1}}.$$

When we multiply that by $|\nabla f|/|\nabla f \cdot p| = |\nabla f|$ from before, the bottom of the fraction cancels out with $|\nabla f|$ and we get just $(8x^2 + 8y^2 - 2)dxdy$ inside our integral.

The last thing we need to do is find bounds for R. Note that our integral,

$$\int\int_R (8x^2 + 8y^2 - 2)dxdy,$$

is ripe for conversion to polar coordinates. R is a disk of radius one, and $8x^2 + 8y^2 - 2$ becomes $8r^2 - 2$ times the extra factor of r. We get

$$\int_0^{2\pi}\int_0^1 (8r^3 - 2r)drd\theta$$

which is equal to 2π.

Example 2: Let S be the portion of the cylinder $y = \ln x$ in the first octant whose projection parallel to the y-axis onto the xz-plane is the rectangle R_{xz}: $1 \le x \le e$, $0 \le z \le 1$. Let n be the unit vector normal to S that points away from the xz-plane. Find the flux of $F = 2y\vec{j} + z\vec{k}$ through S in the direction of n.

Again, following the steps from before:

(1) Our surface S is given by $y = \ln x$, so to rearrange and get f, we might choose $f(x, y, z) = y - \ln x = 0$.
(2) The shadow is given and lies in the xz-plane, so dA is $dxdz$.
(3) A unit vector normal to the shadow is one normal to the xz-plane, so let p be $\vec{j}$.
(4) $\nabla f = -\frac{1}{x}\vec{i} + \vec{j}$.
(5) $|\nabla f| = \sqrt{\frac{1}{x^2} + 1}$.
(6) $|\nabla f \cdot p| = 1$.
(7) $(5)/(6) = (5) = \sqrt{\frac{1}{x^2} + 1}$.

(8) Find the bounds for the region: They have been given to us neatly, so we'll just keep them how they are and most likely not need to change to polar coordinates: $1 \leq x \leq e$, $0 \leq z \leq 1$.

Now we need n. As before,

$$n = \pm\frac{\nabla f}{|\nabla f|} = \pm\frac{-\frac{1}{x}\vec{i} + \vec{j}}{\sqrt{\frac{1}{x^2} + 1}}.$$

Which direction do we choose? We want n to point away from the xz-plane. Our surface f lives above the xz-plane; that is, in the positive y direction. How do we see that? The equation $y = \ln x$ in two dimensions for $1 \leq x \leq e$ has $0 \leq y \leq 1$, and since z is not in the equation, putting the equation into three-dimensional space does not change the relationship between x and y. Therefore our surface lives where y is positive, and to point away from the xz-plane (the $y = 0$ plane), we need the $\vec{j}$ coefficient in n to be positive. Thus we'll take the "plus" in the plus-or-minus n.

[Notice that if we had chosen a different version of f, the version $f = \ln x - y = 0$, we would have the same thing here except that we would have to choose the negative version of n.]

So now we dot F with n, getting

$$F \cdot n = \frac{2y}{\sqrt{\frac{1}{x^2} + 1}}.$$

We have a minor problem here, because dA is $dxdz$ and there's a y in this formula. But recall here that we are working exclusively in the surface $y = \ln x$, so we can replace y by $\ln x$.

$$F \cdot n = \frac{2\ln x}{\sqrt{\frac{1}{x^2} + 1}}$$

When we multiply that by item (7) above, we get back just $2\ln x$.

Thus our total integral is:

$$\int_0^1 \int_1^e 2\ln x dxdz$$

which is equal to 2.